Edge ideals of circulant graphs

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WELCOME TO CLASS!

Project Proposal

This project aims to look at circulant graphs, which can be described as finite simple graphs on which a cyclic group acts transitively on the vertices. Examples of circulant graphs thus include the n-gons, as well as the complete graphs. To every finite graph G on the vertex set $[n] = \{1, 2, ..., n\}$, one can associate a monomial ideal IG in the polynomial ring S = $k[x_1, x_2, ..., x_n]$. This ideal is generated by the monomials x_i and x_j , where $\{i, j\}$ is an edge in G. Such ideals have attracted a lot of attention in commutative algebra for a long time, where researchers have linked algebraic properties of IG to graph-theoretic properties of G. In particular we will study the Betti numbers of IG in the case where G is a circulant graph which is invariant under the dihedral group D_n , and investigate the decomposition of its homologies in terms of the irreducible representations of D_n . This investigation will make use of the computer package Macaulay2 for computing the homologies of the ideals IG.

Research Project | May-July 2023

CHORDAL CIRCULANT GRAPHS

The first steps

Finite simple graphs on which a cyclic group acts transitively on the vertices.

All cycles of length at least four have a chord, separate from the cycle, connecting two vertices within the cycle.

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BETTI TABLES

Sum of the number of connected components of size-i in the compliment graph, minus one!

Counts the basis elements of the linear free resolution in various dimensions ultimately generated by the monomial ideals of the compliment graph.



Input: 1, 2, 3, 4, 5, 6	Input: 1, 2, 3, 4, 5, 6	
$\Sigma = \{\{1, 2, 3, 4, 5, 6\}\}$ and $i = 7$	$\Sigma = \{\{1, 2, 3, 4, 5, 6\}\}$ and $i = 7$	
$ \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{$	$\Sigma = \{\{1, 2, 3, 5, 6\}\}$ and $i = 6$ $V_i = 4 \rightarrow \Sigma = \{\{2, 6\}, \{1, 3, 5\}\}$	PERFEC
$\Sigma = \{ \{3, 5\}, \{2, 4, 6\} \}$ and $i = 5$ $V_i = 5 \rightarrow \Sigma = \{ \{3\}, \{2, 4, 6\} \}$	$\Sigma = \{\{2, 6\}, \{1, 3, 5\}\}$ and $i=5$ $v_i = 6 \rightarrow \Sigma = \{\{2\}, \{1, 3, 5\}\}$	Elimin
$\Sigma = \{ \{5\}, \{2, 4, 6\} \}$ and $i = 4$ $V_i = 3 \rightarrow \Sigma = \{ \{2, 4, 6\} \}$	$\Sigma = \{\{2\}, \{1, 3, 5\}\}$ and $i = 4$ $v_i = 2 \rightarrow \Sigma = \{\{1, 3, 5\}\}$	
$\Sigma = \{ \{2, 4, 6\} \}$ and $i = 3$ $V_i = 2 \rightarrow \Sigma = \{ \{4, 6\} \}$	$\Sigma = \{\{1, 3, 5\}\}$ and $i=3$ $v_i = 1 \rightarrow \Sigma = \{\{3, 5\}\}$	Induced cliques
$\Sigma = \{\{4,6\}\}$ and $i=2$ $V_i = 4 \rightarrow \Sigma = \{\{6\}\}$	$\Sigma = \{ \{3, 5\} \}$ and $i = 2$ $\vee_i = 3 \rightarrow \Sigma = \{ \{5\} \}$	An ordering of vertices sucl induced subgraph.
$\Sigma = \{\{6\}\}$ and $i=1$ $V_i = 6 \rightarrow \Sigma = \{\{\phi\}\}$	$\Sigma = \{ \{ 5 \} \} \text{ and } i = 1$ $\vee_i = 5 \longrightarrow \Sigma = \{ \{ \phi \} \}$	A graph is chordal if and onl has a linear free resolution.
Output: 6, 4, 2, 3, 5, 1	Output: 5,3,1,2,6,4	

TION ORDER

h that, for each vertex, its neighbors form a complete

ly if it has a perfect elimination order. Equivalently, if it

MORE ON PERFECT **ELIMINATION ORDERS**

Algorithm 2.2 from "Minimal free resolutions of linear edge ideals" (Chen, 2010)

Algorithm 2.2. Let *H* be a chordal graph with vertices x_1, \ldots, x_n . Let Σ be a set containing a sequence of sets. Input: $\Sigma = \{\{x_1, ..., x_n\}\}, i = n + 1.$ Step 1: Choose and remove a vertex v from the first set in Σ . Set i := i - 1 and $v_i := v$. If the first set in Σ is now empty, remove it from Σ . Go to setp 2. Step 2: If $\Sigma = \emptyset$, stop. If $\Sigma \neq \emptyset$, suppose $\Sigma = \{S_1, S_2, \dots, S_r\}$. For any $1 \leq j \leq r$, replace the set S_j by two sets T_j and T'_i such that $S_j = T_j \cup T'_i$, $T_j \cap T'_i = \emptyset$, $v_i w \in H$ for any $w \in T_j$ and $v_i w' \notin H$ for any $w' \in T'_i$. Now we set $\Sigma := \{T_1, T_2, \dots, T_r, T'_1, T'_2, \dots, T'_r\}.$ <u>0</u>3 Remove all the empty sets from Σ . Go back to step 1.

Output: v_1, \ldots, v_n .

The algorithm given in Chen (2010) produces the maximum of K! (m!) perfect elimination orderings, in which m is the number of vertices in each disjoint connected component of the compliment, and k is the number of disjoint connected components in the compliment.

(X1, X3, X4 1X2) (Xa, X3, X4 1X5) (Xa, X3, X4 1 X6) (X,1X3, X4, X5)7

Construction 3.4. Let G be a simple graph with vertices x_1, \ldots, x_n such that \overline{G} is chordal. Let x_1, \ldots, x_n be in the reverse order of a perfect elimination order of \overline{G} produced by Algorithm 2.2. If $p \ge 1, q \ge 1, 1 \le i_1 < \cdots < i_p < j_1 < \cdots < j_q \le n$ and $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq pnbhd(x_{j_1})$, then the symbol $(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q})$ will be used to denote the generator of the free S-module $S(-x_{i_1}\cdots x_{i_p}x_{j_1}\cdots x_{j_q})$ in homological degree p+q-1 and multidegree $x_{i_1}\cdots x_{i_p}x_{j_1}\cdots x_{j_q}$. We set

$$\mathcal{B} = \{1\} \cup \bigcup_{p \ge 1, q \ge 1} \left\{ (x_{i_1}, \dots, x_{i_p} \mid x_{j_1}, \dots, x_{j_q}) \colon \begin{array}{l} 1 \le i_1 < \dots < i_p < j_1 < \dots < j_q \le n, \\ \{x_{i_1}, \dots, x_{i_p}\} \subseteq pnbhd(x_{j_1}) \end{array} \right\}.$$

(X, X2, X3, X4 1 X7, X2) (X,, X2, X3, X4 | X5, X0) (X,, X2, X3, X4 | X5, X2) + ~ (X, Xa, X3, X4 1 X6, X0)] x (X1, X2, X3 | X5, X6, X4)] (X1, X2, X4 | Xe, X4, X2) (+" (X,, X3, X4 | X5, X3, X0) (X2, X3, X4 | X5, Xe, Xe) X .. Xa, X3 1 Xe, Xy, Xe) (X,, Xa, Xy | Xs, Xa, Xa) (X,, X3, X4 | X5, Xe, X0) (X2, X3, X4 | X5, X6, X4)

(****************** (+1)*** ***

1×1, ×2, ×3 ×1×1

(+1,+++++++++++++)

1+1'

put: 8,7,6,5,4,3,2,1

Jerse: 12345678

bhd $(x_i) = \phi$

 $nbhd(x_2) = \phi$

public $(X_3) = \phi$

+5

1×00 (+)×10)

(+3' +4) (+3' +6)

pupped (x)=\$

probind (x)=1,2,3,4

probad (x)=1,2,3,4

probad (X)=1,2,3,4

probhd (X)=1,2,3,4

(×11×5) (×31×5)

(X,1X0) (X,1X0)

(X,1X+) (X31X+)

(X,1X0) (X31X0)

(X21X5) (X41X5)

(X21X0) (X41X0)

(X=1×7) (X+1×7)

(XalXo) (XylXo)

Subsets of size 2: (1)(1)=16

×1,1×21×5,×6,×4) (×1,1×21×6,×9,×6) (×1,1×21×5,×6,×6) (×- ×-1× × ×) (×- ×-1× × ×) (×- ×-1× × ×) (×- ×+1× ×) (×- ×+ · X₁, A₂1 X₅, A₆, X₉) (X₁, A₂1 X₆, A₉, X₆) (X₁, A₂1 X₅, A₉, X₆) (X₂, X₃ | X₅, X₆, X₉) (X₂, X₃ | X₆, X₇, X₆) (X₁, A₂1 X₅, A₇, X₆) (X₁, X₂1 X₅, X₆, X₉) (X₂, X₃ | X₅, X₆, X₉) (X₂, X₃ | X₆, X₇, X₆) (X₂, X₃ | X₅, X₇, X₆) (X₂, X₃ | X₅, X₆, X₉) (X₂, X₃ | X₅, X₆, X₉) (X₂, X₃ | X₅, X₇, X₆) (X₂, X₃ | X₅, X₆, X₆) (X₂, X₃ | X₅, X₆, X₉) (X₂, X₁, X₂, X₁, X₂) (X₂, X₃ | X₅, X₆, X₆) $\begin{pmatrix} -x_{e} \\ -x_{s}, X_{4} | X_{2} \end{pmatrix} \begin{pmatrix} (X_{a}, X_{3} | X_{5}, X_{6}, X_{9}) & ((X_{a}, X_{3} | X_{6}, X_{9}, X_{2}) & ((X_{a}, X_{3} | X_{5}, X_{9}, X_{9}) \\ (X_{3}, X_{4} | X_{5}, X_{6}, X_{9}) & (X_{3}, X_{4} | X_{5}, X_{9}, X_{2}) & ((X_{3}, X_{4} | X_{5}, X_{9}, X_{9}) & ((X_{a}, X_{3} | X_{5}, X_{9}, X_{9}) \\ (X_{a}, X_{a} | X_{a}, X_{a} | X_{a}, X_{a}) & (X_{a}, X_{a} | X_{a}, X_{a}, X_{a}) & (X_{a}, X_{a} | X_{a}, X_{a}, X_{a}) \\ (X_{a}, X_{a} | X_{a}, X_{a}, 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(X_{a},X_{4}|X_{5},\chi_{a}) \\ (Y_{a},Y_{a},X_{a},X_{a}) & (Y_{a},X_{a},X_{a}|X_{a},X_{a}) & (X_{a},X_{4}|X_{5},\chi_{6}) & (X_{a},X_{4}|X_{5},\chi_{6}) \\ (X_{a},X_{a},X_{a}|X_{a},X_{a}) & (X_{a},X_{a},X_{a}|X_{a},X_{a}) & (X_{a},X_{a},X_{a}|X_{a},X_{a}) \\ (X_{a},X_{a},X_{a}|X_{a},X_{a}) & (X_{a},X_{a},X_{a}|X_{a},X_{a}) & (X_{a},X_{a},X_{a}) \\ (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) \\ (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) \\ (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) \\ (X_{a},X_{a},X_{a}) & (X_{a},X_{a},X_{a}) & (X_{a$ $\begin{array}{c} (X_{3} | X_{5}, X_{6}, X_{7}, X_{2}) & (X_{1}, X_{2}, X_{3} | X_{5}, X_{6}) & (X_{1}, X_{2}, X_{4} | X_{5}, X_{6}) & (X_{1}, X_{2}, X_{3} | X_{5}, X_{6}) & (X_{1}, X_{2}, X_{4} | X_{5}, X_{6}) & (X_{1}, X_{2}, X_{3} | X_{5}, X_{3}) \\ (X_{1} | X_{5}, X_{6}, X_{7}, X_{2}) & (X_{1}, X_{2}, X_{3} | X_{6}, X_{7}) & (X_{1}, X_{2}, X_{4} | X_{5}, X_{6}) & (X_{2}, X_{3}, X_{4} | X_{5}, X_{6}) \\ (X_{1}, X_{2}, X_{3} | X_{3}, X_{2}) & (X_{1}, X_{2}, X_{4} | X_{5}, X_{3}) & (X_{1}, X_{2}, X_{4} | X_{5}, X_{6}) & (X_{2}, X_{3}, X_{4} | X_{5}, X_{6}) \\ (X_{1}, X_{2}, X_{3}, X_{4} | X_{3}, X_{2}) & (X_{1}, X_{2}, X_{4} | X_{3}, X_{4}) & (X_{2}, X_{3}, X_{4} | X_{5}, X_{6}) \\ (X_{2}, X_{3}, X_{4} | X_{3}, X_{4}) & (X_{2}, X_{3}, X_{4} | X_{3}, X_{4}) & (X_{2}, X_{3}, X_{4} | X_{5}, X_{6}) \\ \end{array}$ $\begin{array}{c} (X_{i},X_{a},X_{3}\,|\,X_{e},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{e},X_{q}) & (X_{i},X_{3},X_{q}\,|\,X_{e},X_{q}) & (X_{a},X_{a},X_{q}\,|\,X_{e},X_{q}) \\ (X_{i},X_{a},X_{3}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{a},X_{3},X_{q}\,|\,X_{e},X_{q}) \\ (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{a},X_{3},X_{q}\,|\,X_{e},X_{q}) \\ (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{a},X_{3},X_{q}\,|\,X_{e},X_{q}) \\ (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{a},X_{a},X_{q}\,|\,X_{e},X_{q}) \\ (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{a},X_{a},X_{q}\,|\,X_{e},X_{q}) \\ (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{a},X_{a},X_{q}\,|\,X_{e},X_{q}) \\ (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) \\ (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) \\ (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) & (X_{i},X_{a},X_{q}\,|\,X_{q},X_{q}) \\ (X_{i},X_{i},X_{i},X_{i}\,|\,X_{q},X_{q}) & (X_{i},X_{i},X_{i}\,|\,X_{q},X_{q}) & (X_{i},X_{i},X_{i}\,|\,X_{q},X_{q}) \\ (X_{i},X_{i},X_{i},X_{i}\,|\,X_{i},X_{q}) & (X_{i},X_{i},X_{i}\,|\,X_{i},X_{q}) & (X_{i},X_{i},X_{i}\,|\,X_{i},X_{q}) \\ (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}\,|\,X_{i},X_{i}) & (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}) & (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}) \\ (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}\,|\,X_{i},X_{i}) & (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}) & (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}) \\ (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}) & (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}) & (X_{i},X_{i},X_{i}\,|\,X_{i},X_{i}) \\ (X_{i},X_{i},X_{i}\,$ $\begin{array}{c} (X_{1}, X_{2}, X_{3} | X_{3}, N_{4}; N_{4}; (X_{1}, X_{2}, X_{4}; N_{4}; N_{4}; N_{4}; N_{3}; N_{4}; N_{4}$ $\begin{array}{c} (X_{i},X_{a},X_{3}|X_{5},X_{4}) & (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{i},X_{3},X_{4}|X_{5},X_{4}) & (X_{i},X_{a},X_{4}|X_{5},X_{4}) \\ (X_{i},X_{a},X_{3}|X_{5},X_{4}) & (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{i},X_{3},X_{4}|X_{5},X_{4}) \\ (X_{i},X_{a},X_{3}|X_{5},X_{4}) & (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{a},X_{3},X_{4}|X_{5},X_{4}) \\ (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{a},X_{3},X_{4}|X_{5},X_{4}) \\ (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{a},X_{3},X_{4}|X_{5},X_{4}) \\ (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{i},X_{a},X_{4}|X_{5},X_{4}) & (X_{i},X_{a},X_{4}|X_{5},X_{4}) \\ (X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) \\ (X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) \\ (X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i},X_{i}) & (X_{i},X_{i},X_{i}) & (X_{i},X_$ $(X_{1}, X_{2}, X_{3}) \times (X_{1}, X_{2}, X_{3}) \times (X_{1}, X_{2}, X_{4}) \times (X_{1}, X_{2}) \times (X_{1}, X_{2}, X_{4}) \times (X_{1}, X_{2}) \times (X_{1}, X_{2}$ $\begin{array}{c} (X_{i}, X_{a}, X_{3}, 1, X_{5}, X_{8}) & (X_{i}, X_{a}, A_{4}, 1, X_{5}, X_{8}) & (X_{i}, X_{3}, A_{4}, 1, X_{5}, X_{8}) & (X_{a}, X_{a}, 1, X_{a}, X_{4}, 1, X_{5}, X_{8}) & (X_{a}, X_{a}, X_{4}, 1, X_{5}, X_{8}) & (X_{a}, X_{3}, X_{4}, 1, X_{5}, X_{8}) & (X_{a}, X_{a}, X_{a}, 1, X_{a}, X_{a}, X_{a}, 1, X_{a}) & (X_{a}, X_{a}, X_{a}, X_{a}, 1, X_{a}) & (X_{a}, X_{a}, X_{a}, 1, X_{a}) & (X_{a}, X_{a}, X_{a}, 1, X_{a}) & (X_{a}, X_{a}, X_{a}, X_{a}, 1, X_{a}) & (X_{a}, X_{a}, X_{a}, X_{a}, X_{a}, X_{a}) & (X_{a}, X_{a}, X_{a}) & (X_{$ (X1, X31X5, X6, X9, X2) (X1, X41 X5, X6, X7, X2) (Xa, X41 X5, X6, X9, Xg) (X,, Xa, X3 1 X5, X+, X0)] (X,, Xa, Xy | Xs, Xc, Xa) (X,, X2, X4 | X5, X6, X4) (3)=4 (X2, X3, X4 | X6, X7, X2)] (X,, Xa, X3 | Xs, Xe, Xe)] (X,, Xa, Xy / X5, X6, Xy) (X1, X3, X4 | X6, X7, X2) Subsets of ... (X2, X3, X4/X

Public (x,)=1,2,3,4 (Xy) = 1, 2, 3, 4

(X,1X3)7 4.2+2.2

 $\binom{4}{4}\binom{4}{1} = 4$

(X1, X3 1X6)

(X21X5,X6)

(X31X5, X6)

(X41X5,X6)

(X1, X41X6)

(X2, X3 1 X5)

(Xe, X3 1 X6)

(Xe, X41X5)

(Xa, X41X6)

(X3, X41X5)

(x,1x4)/

(x,1x,))

(X.1X.)] (X21X3)7

 $(x_{3}|x_{5})]\binom{2}{2}\binom{2}{2}=2$

 $(x_{2}, x_{3}) \times (x_{4}, x_{5}) / (4) / (4) = 4$ $(x_{2}, x_{3}) \times (x_{4}, x_{5}) / (4) / (4) = 4$

 $(X_{4}|X_{5})](2)(2)(2)=2$

y (X1, X2, X3, X4 1 X5) (X1, X2, X4 1 X5, X6)

(X,, X3, X41X5, X6)

(Xa, X3, X4 1X5, X6)

. × 6, × 7, ×2) . (X41X5, X7, X0) · Ke, Xg) (Xy/XS, Xe, Xg)

Subsets of size 7:

(X1, Xa, X3 / X5, X6, X9, X0)]

 $\begin{pmatrix} (X_1, X_3, X_4 | X_5, X_6, X_7, X_2) \\ (X_2, X_3, X_4 | X_5, X_6, X_7, X_2) \end{pmatrix} \begin{pmatrix} (4) \\ (3) = 4 \end{pmatrix}$

(X,, X2, X4 | X5, X6, X7, X0)

(X2, X3, X4 | X5, Xe, X7, X2)

 $\begin{array}{c} (X_{i}, X_{a}, X_{3}, X_{4} | X_{5}, X_{7}, X_{2}) \\ (X_{i}, X_{a}, X_{a}, X_{4} | X_{5}, X_{7}, X_{2}) \end{array} \begin{pmatrix} (4) \\ (3) = 4 \end{pmatrix}$

(X,, Xa, X3, X4 1 × 5, X6, X4) 7

(X,, Xa, X3, X4 1 X6, X4, Xa)

(X1, Xa, X3, X4 | X5, X6, X0)

(X, 1 X3, X4, X5, X6)

(X21X3, X4, X5, X6)

(X1, X2 | X2) (X2 | X4) /

(+3, +4 (×3/×5))

, Subsets of size 5:

1, X2 1X3, X5, X6)

(X1, X2 1 X4, X5, X6)

1, Xa, X31X5, X6)

(tv

(*1, *21×5)

(*2, *31 ×5)

(*3, ×41×5)

(*1, ×31×5)

(X1, X41X5)

1431

(Xa, X41 X5)

subsets of size 3:

[(X,1X5,X6)

(X, 1X0, X+)

(X,1×+,X0)

(X, 1×5,×4)

DUDSEAS OT SIZE 7. (X1, X21X5, X6) (X1, X21X6, X7) (X1, X21X4, X8) (X1, X21X5, X4)

(X11 X21 X51 X6) (X11 X21 X61 X9) (X11 A21 X91 X6) (X11 A21 X91 X6) (X21 X51 X6) (X (X21 X51 X61 X4) (X21 X51 X61 X4) (X21 X51 X61 X4) (X21 X51 X61 X4) (X21 X51 X61 X6) (X21 X6) (X21 X61 X6) (X21 X6)

(X,1×5,×8)

(*,1×0,×0)

[(X2 X5, X6)

(X21X0, X7)

(X2 X7, X8)

(X2 X5, X4)

(X2 X5, X8)

(X2 1×6, ×8)

(XSI X41 X71K8) (XSI X41 ASI ASI (XSI X41 ASI AS) (XSI X41 X61 K8) (XSI X41 X61 K8) (XII X51 X51 X6) (XII X51 X61 X8) (XII X51 X61 K8) (XII X51 X51 X6) (XII X51 X61 K9) (XII X51 X51 X6) (XII X51 X61 X8) (XII X51 X61 K8) (XII X51 X61 K8) (XII X51 X61 K8) (XII X51 X61 X9) (XII X11 X51 X61 X8) (XII X51 X61 K8) (XII X51 K8) (X1, X51X5, X9) (X1, X51X5, X8) (X1, X51X6, X8) (X1, X51X5, X6) (X1, X51X6, X) (X1, X41X5, X8) (X1, X41X6, X8) (X1, X41X5, X6) (X1, X41X6, X4) (X1, X41X6, X9) (X1, X41X6, X9) (X1, X41X6, X9) (X2, X41X6, X9 (X1, X4, X5, X8) (X1, X4, X6, X8) (X1, X4, X5, X6) (X1, X4, X6, X9) (X1, X4, X4, X6, X9) (X1, X4, X4, X6, X9) (X2, X4, X4, X4, X6, X9) (X2, X4, X4, X6, X6) (X2, X4, X4, X6, X6) (X2, X4, X4, X6, X6) (X2, X4, X4, X6) (X2, X4, X6, X6) (X2, X4, X6) (X2, X6) (

 $\frac{1}{2} (X_{1}, X_{2}, X_{3}|X_{3})}{(X_{1}, X_{2}, X_{3}|X_{3})} (X_{1}, X_{2}, X_{3}|X_{3}) (X_{1}, X_{2}, X_{3}|X_{3})}{(X_{1}, X_{2}, X_{4}|X_{3})} (X_{1}, X_{2}, X_{4}|X_{3}) (X_{1}, X_{2}, X$

(X2,X3,X4(X3)) (X2,X3,X4(X2)) (X2,X3,X4(X2))

[(X=1×5,×6)

(X31X0, X7)

(X31×7,×0)

(X31×5,×4)

(X31×5,×8)

(X31×6,×8)

(XylXs,Xe)

(Xy1X0,X+)

(X41 X7, X8)

(X41X5, X4)

(X41×5,×0)

L(X41X6, X8)

not #

(X2, X3 1 X5, X6)

(X,, Xa, X3/X6) (Xa) X4, X3, X6)

(X,, X2, X4/X5)

Subsets of size 6:

(X1, Xa 1 X3, X4, X5, X6)

(X1. Xe, X31 X4, X5, X6)

(X1, Xe, X3, X4 1X5, X6)

(X,1X3, X4, X6))

(Xal X3, X4, X6)

(X,1X4, X5, Xe)

BASIS ELEMENTS OF THE FREE RESOLUTION

Reverse Perfect Elimination Order + Preneighborhoods

THE PROCESS SO FAR

O1 Fill in the Betti Table

i1 : kk = ZZ/101	1 o5 : S-module, quotient of S			
ol = kk	i6 : Mres = M			
ol : QuotientRing	o6 = cokernel x_0x_1 x_1x_2 x_0x_2			
$i2 : S = kk[x_0, x_1, x_2]$	1			
o2 = S	o6 : S-module, quotient of S			
o2 : PolynomialRing	i7 : Ires = res I			
i3 : I = ideal (x_0*x_1,x_1*x_2,x_2*x_0)	1 3 2 o7 = S < S < S < 0			
o3 = ideal (x x , x x , x x) 0 1 1 2 0 2	0 1 2 3			
o3 : Ideal of S	o7 : ChainComplex			
i4 : M = S/I	i8 : betti Ires			
o4 = M	0 1 2 o8 = total: 1 3 2			
o4 : QuotientRing	0: 1 1: . 3 2			
i5 : M = S^1/I	o8 : BettiTally			
o5 = cokernel x_0x_1 x_1x_2 x_0x_2	i9 :			

Construct All Possible "Chen-O2 Symbols" for a given Perfect Elimination Order

Output: 8,7,6,5,4,3,2,1	01234567 <u>n</u>
Reverse: 1 2 3 4 5 6 7 8	0 1
pnbhd (X,) = Ø	1 16 48 68 56 28 8 1
ρ nbhd(X ₂)=Ø	
$pnbhd(x_3) = \emptyset$ 8.4	
pnbhd(x)=Ø	
$pnbnd(x_s) = 1, 2, 3, 4$	4 3 8 4 5 4 5 4
$pnbhd(x_{0}) = 1, 2, 3, 4$	
pnbhd (x2) = 1,2,3,4	Subsets of size 3: $\binom{1}{2}\binom{1}{1} + \binom{1}{1}\binom{1}{2} = 6.4 + 4.6 = 48$
$pnbhd(x_{s}) = 1, 2, 3, 4$	[(X,1X5,X6) [(X31X5,X6) (X1,X21X5) (X1,X21X7)
	$(X_1 X_6, X_7)$ $(X_3 X_6, X_7)$ $(X_2, X_3 X_5)$ $(X_2, X_3 X_7)$
Subsets of size $2: (7)(7) = 16$	(X, I X, 7, X8) (X3 I X, X8) (X3, X4 I X5) (X3, X4 I X3)
(X1 X5) (X3 X5)	$(X_1 X_5, X_4)$ $(X_3 X_5, X_4)$ $(X_1, X_3 X_5)$ $(X_1, X_3 X_7)$
(X1 X6) (X3 X6)	(X1X5,X8) (X3X5,X8) (X1,X4X5) (X1,X4X)
(X, X,) (X3 X,)	$(X_1 X_6, X_8) = (X_3 X_6, X_8) = (X_8, X_4 X_5) = (X_8, X_4 X_7)$
(X1 X8) (X3 X8)	[(Xa1X5,X6) [(X41X5,X6) (X1,X21X6) (X1,X21X8)
(X2 X5) (X4 X5)	$(X_{2} X_{6}, X_{7})$ $(X_{4} X_{6}, X_{7})$ $(X_{2}, X_{3} X_{6})$ $(X_{2}, X_{3} X_{6})$
(X ₂ X ₆) (X ₄ X ₆)	$(X_{2} X_{4},X_{8})$ $(X_{4} X_{4},X_{8})$ $(X_{3},X_{4} X_{6})$ $(X_{3},X_{4} X_{8})$
(X ₄ X ₃) (X ₄ X ₃)	$(X_{2} X_{5}, X_{4})$ $(X_{4} X_{5}, X_{4})$ $(X_{1}, X_{3} X_{6})$ $(X_{1}, X_{3} X_{8})$
$(X_{2} X_{8})$ $(X_{4} X_{8})$	$(X_{2} X_{5},X_{8})$ $(X_{4} X_{5},X_{8})$ $(X_{1},X_{4} X_{6})$ $(X_{1},X_{4} X_{8})$
	$ \lfloor (X_2 X_6, X_8) \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \$
Subsets of size 4:	
(X1, X2 X5, X6) (X1, X2 X6, X7) ($(x_1, x_2, x_3) = (x_1, x_2, x_5, x_4) = (x_1, x_2, x_5, x_8) = (x_1, x_2, x_8)$
(X2, X31 X6, X7) (X2, X31 X7, X8) ($(x_2, x_3 X_5, X_4)$ $(x_2, x_3 X_5, X_8)$ $(x_2, x_3 X_6, X_8)$ $(x_2, x_3 X_5, X_6)$
(X3, X4 X4, X8) (X3, X4 X5, X4) (x3, X4 X5, X8) (X3, X4 X6, X8) (X3, X4 X5, X6) (X3, X4 X6, X7)
(X1, X31X5, X4) (X1, X31X5, X8) (X1, X31 X6, X8) (X1, X31 X5, X6) (X1, X31 X6, X7) (X1, X31 X7, X8)
(X1, X41X5, X8) (X1, X41X6, X8) (X ₁ , X ₄ I X ₅ , X ₆) (X ₁ , X ₄ I X ₆ , X ₇) (X ₁ , X ₄ I X ₇ , X ₈) (X ₁ , X ₄ I X ₅ , X ₇)
(Xa, X41X6, X8) (Xa, X41X5, X6) (Xa, X4IX6, X7) (Xa, X4IX7, X8) (Xa, X4IX5, X7) (Xa, X4IX5, X8)
$(X_1, X_2, X_3 X_5) (X_1, X_2, X_3 X_6) (X_1,$	$X_{R}, X_{3} X_{7}$ (X ₁ , X _R , X ₃ X ₈)
$(X_1, X_2, X_4 X_5)$ $(X_1, X_2, X_4 X_6)$ $(X_1,$	$(x_{a}, x_{4} x_{5}) (x_{1}, x_{a}, x_{4} x_{5}) (\frac{4}{3})(\frac{4}{4}) + (\frac{4}{3})(\frac{4}{2}) + (\frac{4}{3})(\frac{4}{3}) =$
$(X_1, X_3, X_4 X_5)$ $(X_1, X_3, X_4 X_6)$ $(X_1,$	$X_3, X_4 X_3$ (X ₁ , X ₃ , X ₄ X ₆) (3)(1) (a) (a) (1) (3) 4.4 + 6.6 + 4.4 =
(X2, X3, X4 X5) (X2, X3, X4 X6) (X2,	$X_{3}, X_{4} X_{7}$ ($X_{2}, X_{3}, X_{4} X_{1}$)



FORMULA FOR COUNTING BASIS ELEMENTS, AKA BETTI NUMBERS

For a graph on n vertices such that its compliment is the union of two disjoint complete graphs on "/2 vertices, the ith Betti number can be computed as follows: $\sum_{m}^{n/2} \binom{n/2}{k}, \text{ where } m + k = i \text{ and } m, k \ge i - \frac{n}{2}.$

Next Steps

Write out formal proof for my formula, work out and prove other formulas for graphs with compliments comprised of more than two disjoint components, write a code for computing basis elements.

Research Project | May-July 2023



Output: 6, 5, 4, 3, 2	2,1				
Reverse: 1 2 3 4 5	6 0	123	4 5		
$pnbhd(x_i) = \phi$	0 1				
$pnbhd(x_2) = \phi$	1	10 20 15	4 3		
public (x,)=1,2 P	eve = in e		<u>,</u>		
public (x,)=1,2		As 4.	<u> </u>		
public (x,)=1,2,3,4)	· Þ				
public (x,)=1,2,3,4	(⁴ ₂)(⁴ ₁) +	$\begin{pmatrix} 4\\1 \end{pmatrix} \begin{pmatrix} 4\\a \end{pmatrix} = a$	8		(4) (4)
Subsets of size 2:	Subsets of s	size 3:	Subsets o	fsize 4:	(3)(1)
(X,1X3) 4.2+2.2	(X1, X21X3)	(X,1X3,X4) [(X1, X21X3,	X4) 7 (X1, X2, X1	1X6)
(X,1X4)	(X1, X2 X4)	(X,1X3,X5) (X1, X21X3,	×5) (X,, Xs, X,	IX ₅)
$(X, X_5) = \begin{pmatrix} 4\\ 4 \end{pmatrix} \begin{pmatrix} 7\\ 4 \end{pmatrix} = 4$	(X1, X21X5)	(X,1X3,X6	(4) (X1, X21X3,	X6) (X1, X5, X1	1×6)
(X,1X6)	$(X_1, X_2 X_6)$	(X, 1X4, X5	(x1, X21 X4,	X5) (X2, X3, X4	1×5)
(X21X3)]	$(X_1, X_3 X_5)$	(X,1X4,X6	$\begin{pmatrix} 2\\ a \end{pmatrix} (X_1, X_2 X_4, x_4)$	X6) (X2, X3, X4	1×6)
(X21×4) (4)(4)=4	$(X_1, X_3 X_6)$	(X,1×5,×6	(X1, X21X5,	X_) (X,1X3,X4	, X5)
(XaIX5) (4)(1)	$(X_1, X_4 X_5)$	(Xal X3, X4)	(X1, X31X5,	X6) 7 (X2 X3, X4	, X5)
(X ₂ X ₆)	(X,, X41X6)	$(X_2 X_3, X_5)$	(X1, X41X5,	X6) (X,1X3,X5	, X6)
$(X_3 X_5)](2)(2)=2$	$(X_2, X_3 X_5)$	(X2 X3, X6	(X2, X3 1 X5,	X6) (X2 X3,X5	, X6) (4) (4)
(X31X6) (a/(1)	$(X_R, X_S X_6)$	(X21X4, X5) (X _R , X ₄ 1X ₅ ,	X6) (X,1X3, X4	,X6) (1/(3)
$(X_4 X_5)](2)(2)=2$	(X, X41X5)	(XalX4, X6	$\binom{2}{2}\binom{2}{2}(X_3, X_4 X_5)$	X6)] (X2 X3, X4	(x6)
(X41X6) (2/(1) ~	(X2, X41X6)	$(X_a X_5, X_6)$	(X_1, X_2, X_3)	1X5) (X1X4,X5	,×6)
	$(X_3, X_4 X_5)$	(X31×5,X6	(X_1, X_2, X_3)	1X6) (X21X4,X5	,×.)
	(X3, X41X6)	(X41X5,X6) (X_1, X_2, X_4)	X ₅)	
Subsets of size 5:			Subsets of size	6:	
(X1, X2, X5, X4 X5)	(X,, Xa, X41 X5,)	(j)	(X1, Xa X3, X4, X5,)	(₆)	
(X1, X2, X5, X4 X6)	(X,, Xs, X4 X5,)	(j)	(X1, X2, X3 X4, X5,)	(₆)	
(X1, X2 X3, X4, X5)	(Xa, X3, X4 1X5,)	(6)	(X1, X2, X3, X4 X5,)	(₄)	
$(X_1, X_2 X_3, X_5, X_6)$	(X, 1X3, X4, X5,	K6)			
(X1, X2 X4, X5, X6)	(Xal X3, X4, X5,	(6)			
$(X_1, X_2, X_3 X_5, X_6)$					

IS EVERYTHING CLEAR?

