Representations of Alternating Sign and Sign-Restricted Matrices

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Abstract

Alternating sign matrices (ASMs) are square (0,±1)-matrices whose non-zero entries in any row or column alternate, starting and ending with a +1. They first appeared in Dodgson's study of determinants and have since found uses in studying the formation of ice crystals [1]. In combinatorics the enumeration of ASMs of a given size remained an open problem for over a decade [2]. ASMs have been generalised to a class of $m \times n$ (0,±1) matrices known as sign-restricted matrices (SRMs). This report investigates ASMs and SRMs and their representations as monotone triangles and Young tableaux, respectively.

1 Introduction

Alternating sign matrices (ASMs) and sign-restricted matrices (SRMs) are $(0, \pm 1)$-matrices with certain constraints on their partial and total row and column sums. Monotone triangles (MTs) and Young tableaux (YT) are objects of interest in the study of plane partitions in combinatorics. ASMs and SRMs have respective representations as MTs and YT. The aim of my internship was to gain some understanding of each of the mentioned objects and of how they are connected.

Sections 2 and 3 give definitions and aim to establish some intuition for ASMs and SRMs, respectively. Section 4 considers the transformation of ASMs and SRMs into $(0, 1)$-matrices by associating them to their partial-column-sums matrices (PCSMs). I found that the properties of the PCSMs of ASMs and SRMs were key to my understanding of the representations that follow, as such, they are explored in some detail. Section 5 then explains the rpresentation of ASMs as MTs and of SRMs as YT.

2 Alternating Sign Matrices

Key Definition. *An* ***Alternating Sign Matrix*** *(ASM) is an $n \times n$ matrix whose entries are 0, 1 or -1 such that*

- *for $i < n$ the sum of the first i entries in any row (starting from the left) or any column (starting from the top) is 0 or 1. [These summations are referred to as 'partial' row and column sums],*
- *the sum of all the entries in any row or column is 1. [Referred to as 'total' row and column sums].*

The set of all $n \times n$ ASMs will be denoted by $\mathcal{A}_n$. If $A \in \mathcal{A}_n$ then A is said to be an ASM of 'order n'.

The following are examples of alternating sign matrices:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

That the above are ASMs is easily verified by computing all total and partial row and column sums. A more intuitive way to decide whether a square $(0, \pm 1)$-matrix is an ASM is to check if every row and column's non-zero entries alternate in sign, starting and ending with a $+1$. Highlighting the previous examples' $+1$'s in blue and -1 's in orange these alternating properties are easily seen:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Definitions. *Let A be any $m \times n$ matrix.*

*Define the **vertical reflection** of A to be the $m \times n$ matrix formed by swapping A's i^{th} column with its $(n+1-i)^{th}$ column for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.*

*Define the **horizontal reflection** of A to be the $m \times n$ matrix formed by swapping A's j^{th} row with its $(m+1-j)^{th}$ row for all $1 \leq j \leq \lfloor \frac{m}{2} \rfloor$.*

Let A^V denote A's vertical reflection and A^H denote A's horizontal reflection.

For example, let A be the ASM

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then A's vertical and horizontal reflections are, respectively,

$$A^V = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} ; \; A^H = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

These definitions allow us to consider some nice properties of ASMs.

Properties. *Let A be an $n \times n$ ASM.*

i. The first column of A contains exactly one $+1$, all other entries are 0. The first row of A has the same property.

ii. The transpose of A, A^T, is an ASM.

iii. A^V is an ASM.

iv. A^H is an ASM.

The first two properties easily follow from the definition of ASMs. To see why property iii. holds: Let A be an $n \times n$ ASM. Consider the sum of the last j entries in any row of A $(1 \leq j < n)$

$$\begin{aligned} \text{the sum of the last } j \text{ entries} &= 1 - (\text{ the sum of the first } n-j \text{ entries }) \\ &= 1 - (\, 0 \text{ or } 1\,) \\ &= 0 \text{ or } 1. \end{aligned}$$

So the rows of an ASM must have the same alternating property whether they are read from the left or from the right. This shows A^V is an ASM, as required.

Property iv. holds by a composition of properties ii. and iii. [$A^H = ((A^T)^V)^T$].

These properties are linked to the study of symmetry classes of ASMs. For example, for n odd, it is interesting to ask how many $A \in \mathcal{A}_n$ are such that $A = A^V$. For the answer to this question see [3]. For recent results in this field see [4].

2.1 Permutation Matrices

One important subset of the set of $n \times n$ ASMs are the permutation matrices.

Definition. *A* ***permutation matrix*** *is an $n \times n$ (0,1)-matrix with exactly one 1 in every row and every column.*

Let $\mathcal{P}_n$ denote the set of $n \times n$ permutation matrices.

The following are examples of permutation matrices of various sizes:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Properties. *Let A be an $n \times n$ ASM.*

- *A is a permutation matrix if and only if it contains no -1's.*
- *A must contain at least n non-zero entries.*

$\mathcal{P}_n$ is exactly that subset of $\mathcal{A}_n$ whose elements have the smallest number of non-zero entries.

2.2 Diamond ASMs

Another interesting subset of ASMs are the 'diamond ASMs'. There are various definitions, but for this report:

Definition. *The* ***diamond ASMs*** *of order n are those elements of $\mathcal{A}_n$ with the largest number of non-zero entries.*

For odd n, the diamond ASM of order n is unique. Below are the 3×3 and 5×5 diamond ASMs.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

For even n, there are two distinct diamond ASMs that are vertical (and horizontal) reflections of each other. Below are both diamond ASMs of order 4:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

3 Sign-Restricted Matrices

Alternating sign matrices have been generalised to another class of $(0, \pm 1)$-matrices known as sign-restricted matrices.

Key Definition. *A **Sign-Restricted Matrix** (SRM) is an $m \times n$ matrix whose entries are 0, 1 or -1 such that*

- *all partial and total column sums are 0 or 1,*
- *all partial and total row sums are non-negative.*

The set of all $m \times n$ SRMs will be denoted $\mathcal{S}_{m,n}$.

The following are examples of sign-restricted matrices.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

Let A' be an $m \times n$ SRM.

Non-zero entries in columns of A' must alternate between $+1$ and -1 starting with a $+1$. Columns of A' don't necessarily end in a $+1$. ${A'}^H$ is not necessarily an SRM.

In any row of A', the first j entries contain at least as many $+1$'s as -1's, for all $1 \leq j \leq n$. ${A'}^T$ and ${A'}^V$ are not necessarily SRMs.

3.1 Normalized SRMs

Definition. *A **normalized SRM** (nSRM) is a sign-restricted matrix with all total column sums equal to 1.*

Any SRM can be easily transformed into a normalized SRM by the addition of extra rows each containing one $+1$ in a column previously ending -1 and all other entries as 0.

For example, the two SRMs above can be normalized as follows

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The horizontal reflection of an nSRM is an nSRM.

3.2 Special SRMs

Definition. *A **special SRM** (sSRM) is a normalized sign-restricted matrix with all total row sums equal to 1.*

Special SRMs must be square. A justification of this is given in section 4.2. (**Note.** Not every square SRM is an sSRM).

For $n \geq 5$ there exist $n \times n$ special SRMs which can't be obtained by the permutation of columns of any element of $\mathcal{A}_n$. For example:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

4 Partial-Column-Sums Matrices

Key Definition. *The **Partial-Column-Sums Matrix** (PCSM) of any $m \times n$ matrix, A, is an $m \times n$ matrix, B, which has as its ij-entry the sum of the first i entries in column j of A.*

Worked Example 4.0.1. *A_1 is an example of an ASM. B_1 is A_1's PCSM. The highlighted entry of B_1 is the sum of the entries highlighted in A_1.*

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & \mathit{1} & 0 & 0 \\ 0 & 1 & 0 & \mathit{-1} & 1 & 0 \\ 1 & 0 & 0 & \mathit{0} & -1 & 1 \\ 0 & 0 & 0 & \mathit{1} & 0 & 0 \\ 0 & 0 & 1 & \mathit{-1} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \leftrightarrow B_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & \mathit{0} & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Worked Example 4.0.2. *A_2 is an SRM. B_2 is A_2's PCSM. The highlighted entry of B_2 is the sum of the entries highlighted in A_2.*

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & \mathit{1} \\ 0 & 0 & 1 & 1 & -1 & \mathit{-1} \\ 0 & 1 & 0 & -1 & 1 & \mathit{1} \\ 0 & 0 & 0 & 1 & -1 & \mathit{0} \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \leftrightarrow B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & \mathit{1} \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The PCSMs of ASMs and SRMs are useful because they transform $(0, \pm 1)$-matrices into $(0,1)$-matrices. This follows from the partial and total column sums of ASMs and SRMs being 0 or 1.

4.1 PCSMs of ASMs

This section considers the properties of the PCSMs of ASMs.

Lemma 4.1.1. *There are exactly k 1's in row k of the PCSM of an $n \times n$ ASM $(1 \leq k \leq n)$.*

Proof. Let A be an $n \times n$ ASM and B be A's PCSM. By swapping the order of summation,

$$\text{(the sum of row } k \text{ of } B) = \text{(the sum of the first } k \text{ rows of } A), \quad 1 \leq k \leq n.$$

But any row of an ASM sums to 1, so,

$$\text{(the sum of the first k rows of } A) = \text{k}, \quad 1 \leq k \leq n.$$

The result follows from the fact that B is a $(0,1)$-matrix. $\square$

Let $A = [a_{ij}]$ be an $n \times n$ ASM and $B = [b_{ij}]$ be its PCSM. The column vector $(0 \quad 1)^T$ 'appears' in B, specifically,

$$\begin{pmatrix} b_{ij} \\ b_{i+1,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{for some } 1 \leq i < m, 1 \leq j \leq n$$

if and only if $a_{i+1,j} = 1$. Similarly, the appearance of $(1 \quad 0)^T$ in B corresponds to a -1 in A, and the appearance of $(0 \quad 0)^T$ or $(1 \quad 1)^T$ correspond to 0's in A.

Ignoring 0's, any row of A reads $\begin{pmatrix} 1 & -1 & 1 & \dots & -1 & 1 \end{pmatrix}$. So, <u>every</u> matrix consisting of two consecutive rows of B (in their original order), on removing columns with repeated entries, becomes

$$Z = \begin{pmatrix} 0 & 1 & 0 & \dots & 1 & 0 \\ 1 & 0 & 1 & \dots & 0 & 1 \end{pmatrix}.$$

This property of B is referred to as the **zig-zag property**. Any $2 \times c$ matrix (c odd), which has the same form as Z is referred to as a **zig-zag matrix**.

Worked Example 4.1.2. *Let A be the given 5×5 ASM, then B is its PCSM.*

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \leftrightarrow B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

First observe that B has k 1's in row k ($1 \leq k \leq 5$). Next, consider <u>all</u> sub-matrices consisting of an (ordered) pair of consecutive rows in B:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Deleting columns with repeated entries leaves:

$$\begin{pmatrix} 0 & & 1 & 0 & \\ 1 & & 0 & 1 & \end{pmatrix}; \begin{pmatrix} & & 0 & 1 & 0 \\ & & 1 & 0 & 1 \end{pmatrix}; \begin{pmatrix} & 0 & 1 & 0 & \\ & 1 & 0 & 1 & \end{pmatrix}; \begin{pmatrix} & & 0 & & \\ & & 1 & & \end{pmatrix}.$$

Which are all zig-zag matrices. So B has the zig-zag property.

The combination of Lemma 4.1.1 and the zig-zag property gives a complete characterisation of the PCSMs of ASMs. Formal statement and proof can be found in [5].

A characterisation of the PCSMs of permutation matrices is also given in [5].

Let B be the PCSM of an $n \times n$ diamond ASM. In pairs of consecutive rows of B, the zig-zag matrix must appear in a set of consecutive columns. Also, the middle column of these zig-zag sub-matrices must appear in the middle column of B for n odd, or in column $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ of B for n even.

Worked Example 4.1.3. *Let A be a 4×4 diamond ASM and B be A's PCSM,*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Then the three sub-matrices formed from pairs of consecutive rows in B are:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and have the properties described.

4.2 PCSMs of SRMs

Now turn to the PCSMs of SRMs.

Lemma 4.2.1. *The number of 1's in each row of the PCSM of an $m \times n$ SRM is weakly increasing from row 1 to row m.*

Proof. Let A' be an $m \times n$ SRM and B' be its PCSM, for $1 \leq k \leq m$,

$$\begin{aligned}(\text{the sum of the } k^{th} \text{ row of } B') &= (\text{the sum of the } (k-1)^{st} \text{ row of } B') \\ &+ (\text{the sum of the } k^{th} \text{ row of } A')\end{aligned}$$

if (the sum of the 0^{th} row of B) $:= 0$. Since the total row sums of A' are non-negative,

$$(\text{the sum of the } k^{th} \text{ row of } B') \geq (\text{the sum of the } (k-1)^{st} \text{ row of } B').$$

Since B' a $(0,1)$-matrix the result follows. □

ASMs and SRMs having similar restrictions on their partial column sums. So the entries of A' correspond to 2×1 vectors in B' in a similar way to those of ASMs and their PCSMs in section 4.1:

$$(0 \quad 1)^T\text{'s in } B' \text{ correspond to 1's in } A' \text{ ; } (1 \quad 0)^T\text{'s in } B' \text{ correspond to } -1\text{'s in } A' \text{ ;}$$

$$(0 \quad 0)^T \text{ or } (1 \quad 1)^T \text{ in } B' \text{ correspond to 0's in } A'.$$

Since partial row sums of A' are non-negative, the first j columns of any sub-matrix consisting of pairs of consecutive rows in B' must contain at least as many $(0 \quad 1)^T$'s as $(1 \quad 0)^T$'s, for all $(1 \leq j < m)$.

Worked Example 4.2.2. *Let A' be the given 5×6 SRM and B' be its PCSM,*

$$A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \leftrightarrow B\,' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Then the four sub-matrices formed from pairs of consecutive rows of $B\,'$ are:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

In any of these matrices, counting (from the left) both the $(0 \quad 1)^T$ and $(1 \quad 0)^T$ columns, the count of the former must always equal or exceed that of the latter.

PCSMs of nSRMs additionally have the property that their last row must consists entirely of 1's.

PCSMs of sSRMs have the property that row k must contain exactly k 1's. Proof is analogous to that of Lemma 4.1.1. Since sSRMs are normalized by definition their PCSMs must be square. It follows that sSRMs are themselves square.

Given an $m \times n$ $(0,1)$-matrix, B', an $m \times n$ $(0,\pm 1)$-matrix, A', can be constructed by letting the i^{th} row of A' be equal to

$$(\text{the } i^{th} \text{ row of } B') - (\text{the } (i-1)^{st} \text{ row of } B'), \quad \text{for } 1 \leq i \leq m,$$

where (the 0^{th} row of B') $:=$ (the zero vector of length n). Due to this, bijections between subsets of $\mathcal{S}_{m,n}$ and classes of $(0,1)$-matrices with the given properties can be established. [5] contains formal statement and proof for the case of nSRMs.

5 Monotone Triangles and Young Tableaux

More surprising representations of ASMs and SRMs are as monotone triangles and Young tableaux, respectively.

5.1 ASMs $\leftrightarrow$ Monotone Triangles

Key Definition. *A* ***Monotone Triangle*** *(MT) of order n is a triangular array with entries from $\{1, 2, \ldots, n\}$ and n entries along each side, such that*

- *entries strictly increasing along rows (left to right),*
- *entries weakly increasing along both ascending ($\nearrow$) and descending ($\searrow$) diagonals.*

The following is an example of a monotone triangle

$$\begin{array}{ccccccccc} &&&&2&&&& \\ &&&1&&3&&& \\ &&1&&3&&4&& \\ &1&&2&&4&&5& \\ 1&&2&&3&&4&&5 \end{array}.$$

Let A be a given ASM, a corresponding monotone triangle, T, can be constructed as follows:

1. Construct A's PCSM, B.
2. Replace all the 1's in B with their column numbers, to obtain a new matrix, C.
3. Left-justify the non-zero entries of C to get a lower triangular matrix, L_T.
4. Arrange non-zero entries of L_T into a triangular array, T.

Worked Example 5.1.1. *Here is the step-by-step construction of monotone triangle, T, from a given ASM, A.*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \leftrightarrow \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\leftrightarrow \quad C = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 4 & 0 \\ 1 & 2 & 0 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad \leftrightarrow \quad L_T = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 & 0 \\ 1 & 2 & 4 & 5 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\leftrightarrow \quad T = \begin{array}{ccccccccc} &&&&2&&&& \\ &&&1&&3&&& \\ &&1&&3&&4&& \\ &1&&2&&4&&5& \\ 1&&2&&3&&4&&5 \end{array}.$$

That T is a monotone triangle in general follows from the characterisation of the PCSM of an ASM from section 4.1. In particular:

- that T is a triangle follows from B having k 1's in row k;

- strict increase along T's rows is due to the construction of C from B;
- weak increase on ascending and descending diagonals follows from B having the zig-zag property.

Going step-by-step, the process of constructing an MT from an ASM is easily reversed. This gives the well known bijection between elements of $\mathcal{A}_n$ and monotone triangles of order n.

5.2 SRMs $\leftrightarrow$ Psuedo Monotone Triangles

The transformation of an ASM into a MT can be nicely generalised to any $m \times n$ SRM, A', by blindly applying the first three steps of the same method, namely:

1. Construct A' 's PCSM, B'.
2. Replace all the 1's in B' with their column numbers to get a new matrix, C'.
3. Left justify the non-zero entries of C' to obtain some $m \times n$ matrix, L'.

Since SRMs don't, in general, have total row sums equal to 1, L' isn't necessarily lower triangular. So it no longer makes sense to place entries into a triangular array.

Worked Example 5.2.1. *Method of constructing MTs from ASMs applied to an SRM, A'.*

$$A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \quad \leftrightarrow \quad B' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\leftrightarrow \quad C' = \begin{pmatrix} 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 3 & 4 & 5 & 0 \\ 0 & 2 & 3 & 0 & 5 & 6 \\ 0 & 2 & 3 & 4 & 0 & 6 \\ 1 & 2 & 0 & 4 & 5 & 6 \end{pmatrix} \quad \leftrightarrow \quad L' = \begin{pmatrix} 5 & 6 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 & 0 \\ 2 & 3 & 5 & 6 & 0 & 0 \\ 2 & 3 & 4 & 6 & 0 & 0 \\ 1 & 2 & 4 & 5 & 6 & 0 \end{pmatrix}$$

Matrices L' constructed in this way must satisfy the following definition:

Key Definition. *A **Pseudo Monotone Triangle** (PMT) is an $m \times n$ matrix with entries from $\{0, 1, \ldots, n\}$ whose non-zero entries are left-justified and are such that,*

- *the number of non-zero entries in each row is weakly increasing (from row 1 to row m),*
- *non-zero entries strictly increase along each row (from left to right),*
- *the non-zero entries weakly increase up each column (from bottom to top).*

That L' is a PMT follows directly from the properties of the PCSMs of SRMs given in section 4.2, specifically,

- the number of non-zero entries in each row weakly increase is due to Lemma 4.2.1;
- non-zero entries strictly increase along the rows by construction of C;
- that the non-zero entries weakly decrease down any column is due to B's pairs of consecutive rows always having at least as many $(0 \quad 1)^T$'s as $(1 \quad 0)^T$'s (counting from left to right) .

By reversing the method a bijection between $\mathcal{S}_{m,n}$ and the set of $m \times n$ PMTs can be established.

The set of $m \times n$ nSRMs correspond to those $m \times n$ PMTs whose last row contains all n entries.

The set of $n \times n$ sSRMs correspond to the set of $n \times n$ PMTs that are lower triangular matrices without 0's on or below the main diagonal.

5.3 SRMs $\leftrightarrow$ Young Tableaux

It is easy to set up a bijection between PMTs and Young tableaux, objects studied in combinatorics, also known as semistandard Young tableaux or column-strict reversed plane partitions.

> **Key Definition.** *A* ***Young Tableau*** *(YT) is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row (top to bottom) and filled with positive integers, such that*
>
> - *entries weakly increase along each row;*
> - *entries strictly increase down each column.*

Y is an example of a Young tableau,

$$Y = \begin{array}{llllll} \boxed{1} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} \\ \boxed{2} & \boxed{3} & \boxed{3} & \boxed{4} & \boxed{6} & \\ \boxed{4} & \boxed{4} & \boxed{5} & & & \\ \boxed{5} & \boxed{6} & \boxed{8} & & & \\ \boxed{8} & & & & & \end{array} .$$

Once a PMT, L', of an $m \times n$ SRM has been constructed a YT, T', with at most m columns and entries in $\{1, 2, \dots, n\}$ is easily obtained by rotating L' 90° clockwise, removing its zeros and placing boxes around the remaining entries.

This process can again be easily reversed to set up bijections as long as you are careful about defining the size of the PMT you are trying to reach and only add necessary rows or columns of zeros to the top or to the right of the PMT.

Consider again the 5×6 PMT from Example 5.2.1, $L'_{5.2.1}$, its corresponding YT, $T'_{5.2.1}$, is then

$$L'_{5.2.1} = \begin{pmatrix} 5 & 6 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 & 0 \\ 2 & 3 & 5 & 6 & 0 & 0 \\ 2 & 3 & 4 & 6 & 0 & 0 \\ 1 & 2 & 4 & 5 & 6 & 0 \end{pmatrix} \quad \leftrightarrow \quad T'_{5.2.1} = \begin{array}{lllll} \boxed{1} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{5} \\ \boxed{2} & \boxed{3} & \boxed{3} & \boxed{4} & \boxed{6} \\ \boxed{4} & \boxed{4} & \boxed{5} & \boxed{5} & \\ \boxed{5} & \boxed{6} & \boxed{6} & & \\ \boxed{6} & & & & \end{array} .$$

The fact that T''s entries have the properties of a YT follow directly from the restrictions on the entries of L'. Remaining is only the restrictions on the 'shape' of T'.

Unfortunately, the restriction on a weakly increasing number of non-zero entries in the rows of L' corresponds to weakly a decreasing number of boxes in the columns of T' (instead of in its rows).

Definitions. *Let Y be a Young tableau*

The ***shape*** *of Y is a list of its row lengths (top to bottom). It corresponds to a partition of the total number of boxes.*

The ***diagram****, λ, of Y is just Y's boxes with entries removed.*

The ***conjugate*** *of a diagram, $\tilde{\lambda}$, is the diagram obtained from flipping λ over its 'main diagonal'.*

For example, Y above has shape (6, 5, 3, 3, 1) corresponding to a partition of 18, ($18 = 6 + 5 + 3 + 3 + 1$). It has diagram, λ_Y, which has conjugate, $\tilde{\lambda}_Y$, shown below:

$$\lambda_Y = \begin{array}{llllll} \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \\ \boxed{} & \boxed{} & \boxed{} & & & \\ \boxed{} & \boxed{} & \boxed{} & & & \\ \boxed{} & & & & & \end{array} \quad ; \quad \tilde{\lambda}_Y = \begin{array}{lllll} \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} & \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} & \\ \boxed{} & \boxed{} & & & \\ \boxed{} & \boxed{} & & & \\ \boxed{} & & & & \end{array} .$$

If a diagram λ corresponds to a partition of $n \in \mathbb{N}$, then so too does its conjugate, $\tilde{\lambda}$ (see [6]), i.e. if λ has a weakly decreasing number of boxes in its rows, then $\tilde{\lambda}$ does too.

Let $\lambda_{T'}$ be the diagram of T', by the restriction on its columns, its conjugate, $\tilde{\lambda}_{T'}$ has weakly decreasing numbers of boxes per row. Taking the conjugate again returns you to the original diagram, $\lambda_{T'}$. So $\lambda_{T'}$ has a weakly decreasing number of boxes per row, as required.

A more direct bijection between $\mathcal{S}_{m,n}$ and the set of Young tableaux with at most m columns and entries from $\{1, 2, \ldots, n\}$ is given in [7]. (**Note.** In [7] a different notation for Young tableaux is used, essentially a 'horizontal reflection' of the notation used here).

The set of $m \times n$ nSRMs is in bijection with YT that have exactly n rows and at most m columns. Their shapes are partitions of integers between n and $m \cdot n$ (inclusive) with exactly n parts and largest part at most m.

The set of $n \times n$ sSRMs is in bijection with YT that have exactly n rows and n columns. They have shape $(n,\, n-1,\, \ldots,\, 2,\, 1)$. Their shapes are partitions of the n^{th} triangular number.

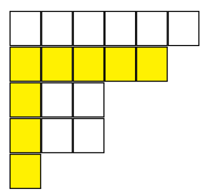
5.4 Enumeration of classes of SRMs

The methods developed for counting Young tableaux can be used directly to count classes of SRMs. The box in the i^{th} row and the j^{th} column of a YT's diagram is denoted (i, j).

Definition. *For λ the diagram of a Young tableau, define the **hook length** of $(i, j) \in \lambda$, $h(i, j)$, to be*

$$\begin{aligned} h(i,j) \;&=\; 1 + \textit{(the number of boxes to the right of box } (i,j) \textit{ in its row)} \\ &\quad\; + \textit{(the number of boxes below box } (i,j) \textit{ in its column)} \end{aligned}$$

Worked Example 5.4.1. *Consider again λ_Y. To determine h(2,1) simply count the highlighted cells:*

$\lambda_Y =$ 

So h(2,1) = 8. Similarly, h(1,3) = 7, h(3,3) = 2 and h(4,3) = 1.

The number of YT on a diagram λ with entries from $\{1, 2, \ldots, n\}$ is given by

$$\prod_{(i,j)\in\lambda} \frac{n+j-i}{h(i,j)}.$$

Unfortunately, the proof of this formula was beyond the scope of my research, but details or where to find them are given in [6].

The size of $\mathcal{S}_{m,n}$ is

$$1 + \sum_{\lambda} \prod_{(i,j)\in\lambda} \frac{n+j-i}{h(i,j)}$$

where the summation is over all λ whose shapes are partitions of integers from 1 to $m \cdot n$ with at most n parts, and largest part at most m.

The number of $m \times n$ nSRMs is given by

$$\sum_{\lambda} \prod_{(i,j)\in\lambda} \frac{n+j-i}{h(i,j)}$$

where the summation is over all λ whose shapes are partitions of the integers from n to $m \cdot n$ with exactly n parts and largest part at most m.

The set of $n \times n$ sSRMs has size

$$\prod_{(i,j)\in\lambda} \frac{n+j-i}{h(i,j)}$$

where λ is the diagram of the partition $(n,\, n-1,\, \dots,\, 2,\, 1)$ of the n^{th} triangular number.

Worked Example 5.4.2. *To calculate the number of 5×5 sSRMs consider their diagram, λ*

$$\lambda = \ydiagram{5,4,3,2,1}$$

Listing the value of $n+j-i$ and the hook length $h(i,j)$ for each $(i,j) \in \lambda$, in separate copies of λ

$$\begin{ytableau} 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 \\ 3 & 4 & 5 \\ 2 & 3 \\ 1 \end{ytableau} \quad , \quad \begin{ytableau} 9 & 7 & 5 & 3 & 1 \\ 7 & 5 & 3 & 1 \\ 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau}$$

Then the number of 5×5 sSRMs $= \frac{\textit{the product of the numbers in the first diagram}}{\textit{the product of the numbers in the second diagram}} = \frac{4572288000}{4465125} = 1024.$

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