Calculation of Self and Mutual Impedances in Planar Magnetic Structures

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Abstract—The high frequency operation of magnetic components, in applications such as filters, makes them ideal candidates for thick film technology along with resistors and capacitors. This in turn leads to distinct advantages over labor intensive wire wound components: improved reliability, repeatability, accuracy and consequential cost reductions. This paper establishes a new set of formulas for the self and mutual impedances of planar coils on ferromagnetic substrates. A planar coil in air is a special case of the generalized formulas. The formulas are derived directly from Maxwell's equations and therefore serve as a useful yardstick for simpler approximations. The formulas take full account of the current density distribution in the coil cross-section and the eddy current losses in the substrate. Experimental and calculated impedances up to 100 MHz are presented for a four layer device with three turns per layer which is 150 µm thick and 40 mm² in area.

Abbreviations

- \( a, r \): filament radii, see Fig. 1.
- \( C \): interlayer capacitance.
- \( d \): dielectric thickness above ferromagnetic substrate.
- \( d_1, d_2 \): height of filaments or coil centers above ferromagnetic substrate.
- \( h_1, h_2 \): coil heights in axial direction.
- \( J(r) \): current density at radius \( r \).
- \( J_n(x) \): Bessel function of the first kind, order \( n \).
- \( K(f), E(f) \): Complete Elliptic Integrals of the first and second kind respectively.
- \( L_1 \): self inductance of coil 1 in air.
- \( L_s \): additional coil inductance due to substrate.
- \( M \): mutual inductance between two filaments in air.
- \( M_{12} \): mutual inductance between two coils.
- \( Q, S \): defined in equations (13) and (14).
- \( r_n \): geometric mean, \( GM = \sqrt{(r_1 r_2)} \).
- \( R_i \): additional coil resistance due to substrate.
- \( R_{dc} \): coil dc resistance.
- \( z \): axial separation.
- \( Z_s \): additional mutual impedance due to presence of ferromagnetic substrate.
- \( \omega \): angular frequency (rad/s).
- \( \sigma \): electrical conductivity of substrate.
- \( \mu_0 \): permeability of free space \((4\pi \times 10^{-7}) \) H/m.
- \( \mu_r \): relative permeability of the substrate.
- \( \phi, \eta \): defined in equations (21) and (22).

I. INTRODUCTION

The momentum towards high density electronic circuits continues unabated. The effects are obvious in very large scale integration (VLSI) design: component densities are being quadrupled every three years. In the case of magnetic components, modern microelectronic techniques such as thick film and thin film technologies are being examined with a view to reducing size and cost and to improving reliability. Planar magnetic components can become an integral part of the process, whereby resistors and capacitors are already established components. One of the major drawbacks in establishing planar magnetic technology is the lack of accurate analytical models for the type of structures encountered. Prototypes are expensive to fabricate and test. One would normally expect to complete a second cycle of fabrication and testing before a final design is achieved. While this procedure may give a better insight, it does not lead to an established design methodology. The purpose of this paper is to address this situation.

The starting point for all inductance calculations is the celebrated formula for the mutual inductance between two filaments given by Maxwell [1]. Coils have a finite cross-section and the standard technique is to integrate the filamentary formula over the cross-section, assuming a constant current density. Alternatively, an approximate result can be obtained by placing a filament at the center of each coil and the mutual inductance can be calculated directly from the filament formula. These approaches have worked well in the past [2]. In the case of planar magnetic components, the aspect ratio of height to width of a section is usually very severe. This paper shows that the current density is not constant and when this factor is taken into account, accuracy is greatly improved.

The simplest configuration of a planar magnetic component is the air-cored spiral inductor [3]. Despite its physical simplicity, it forms the basis for more advanced
configurations such as magnetic substrates [4] and sandwich inductors [5]. This paper establishes a new formula for the mutual inductance between two planar spirals in air which takes full account of the current density distribution in the planar section. The result can be extended to a component with several turns per layer and with several layers. The next step is to add a magnetic substrate, which introduces eddy current losses. A frequency dependent mutual impedance formula for this case is derived, which takes the eddy current losses into account.

Planar magnetic components are suitable because of their small size. This is a direct manifestation of the general principle that the size of magnetic components is reduced as frequency increases. Unfortunately high frequency operation gives rise to unwanted skin effect and proximity effect losses. In multilayer devices the interlayer capacitance introduces resonance at high frequencies. Experimental results are compared with predicted values for a 4 layer spiral inductor with 3 turns per layer, measurements are taken up to 100 MHz.

II. SPIRAL COILS IN AIR

The derivation of the general mutual inductance formula for planar structures starts with the mutual inductance between two filaments [1].

\[ M = \mu_0 \pi ar \sum_{0}^{a} J_1(kr)J_1(ka)e^{-k|z|} dk \] (1)

where \( J_1 \) is a Bessel function of the first kind, \( a, r \) are the filament radii shown in Fig. 1 and \( \mu_0 \) is the permeability of free space.

The solution of (1) can be written in terms of elliptic integrals

\[ M = \mu_0 \sqrt{ar} \frac{2}{f} \left[ \frac{f^2}{2} K(f) - E(f) \right] \] (2)

where \( K(f) \) and \( E(f) \) are complete elliptic integrals of the first and second kind, respectively and where

\[ f = \frac{4ar}{\sqrt{z^2 + (a + r)^2}}. \] (3)

Fig. 2 shows the arrangement and dimensions of two illustrative circular and concentric planar sections. In practice a spiral arrangement would connect two sections in series, which can be accurately modelled by the concentric circular coils. The traditional approach involves integrating the filamentary formula (1) over each cross-section, assuming the current density is constant in each section [2], [6]–[8]. The approach works well when the width to height ratio of the section approaches 1. However in a planar structure this ratio could be 50:1. Evidently the path on the inside edge of the section is much shorter than that on the outside edge and therefore the resistance is reduced on the inside, with consequential higher current density. It is reasonable then to assume that an inverse relationship exists between the current density \( J(r) \) and the radius \( r \). Since the height of the section is much smaller than the width we shall assume that there is negligible variation in current density in the \( z \) direction.

Given that the total current in the section is \( I \), then

\[ h \int_{r_1}^{r_2} J(r) \, dr = I \] (4)

\[ J(r) = \frac{K}{r} \] (5)

Solving (4) and (5) gives

\[ J(r) = \frac{I}{h.r. \ln \left( \frac{r_2}{r_1} \right)} \] (6)

In the following analysis, the current is sinusoidal,

\[ J_d(r, t) = J(r) e^{j\omega t} \] (7)

where \( \omega \) is the angular frequency.

The voltage induced in a filament at \( (r, \tau) \) in coil 1 due to the current in an annular section \( da \times dr_2 \) at radius \( a \)
in coil 2 is

$$dV = j\omega M(a) \, da \, d\tau_2$$  \hspace{1cm} (8)$$

where $M$ is the mutual inductance between the filaments at $(r, \tau_1)$ and $(a, z + \tau_2)$. The total voltage at $(r, \tau_1)$ due to all the current in coil 2 is obtained by integrating (8) over the cross-section of coil 2

$$V(r) = j\omega \mu_0 \pi \int_0^\infty \int_{-h_1/2}^{h_1/2} a I(kr) J_1(ka) e^{-k|z + \tau_2 - \tau_1|} \, da \, d\tau_2 \, dk.$$  \hspace{1cm} (9)$$

The power transferred to the annular segment at $(r, \tau_1)$ due to coil 2 is

$$dP = V(r) J(r) \, dr \, d\tau_1.$$  \hspace{1cm} (10)$$

Finally, the total power transferred to coil 1 is found by integrating (10) over its cross-section

$$P = j\omega \mu_0 \pi \int_0^\infty \int_{-h_1/2}^{h_1/2} a I(kr) J_1(ka) e^{-k|z + \tau_2 - \tau_1|} \, da \, d\tau_1 \, d\tau_2 \, dk.$$  \hspace{1cm} (11)$$

The internal integrals are readily solved, with the aid of (6), to give

$$P = j\omega \mu_0 \pi \frac{I_1 I_2}{h_1 \ln \left( \frac{r_2}{r_1} \right) h_2 \ln \left( \frac{a_2}{a_1} \right)} \int_0^\infty \frac{1}{k} \left[ S(kr_1, kr_2) S(ka_1, ka_2) Q(kh_1, kh_2) e^{-k|z|} \right] \, dk$$  \hspace{1cm} (12)$$

where

$$Q(kx, ky) = \frac{2}{k} \left[ \cosh k \frac{x + y}{2} - \cosh k \frac{x - y}{2} \right]$$

$$z > \frac{h_1 + h_2}{2}$$

$$z = \frac{\frac{h_1 + h_2}{2}}{h + e^{4k} - \frac{h_1 + h_2}{2}}$$

$$S(kx, ky) = \frac{J_0(kx) - J_0(ky)}{k}$$  \hspace{1cm} (13)$$

But

$$P = \nu_2 I_2 = j\omega M_{12} I_1 I_2$$  \hspace{1cm} (15)$$

where $M_{12}$ is the mutual inductance between the two coils. Equating (12) and (15):

$$M_{12} = \frac{\mu_0 \pi}{h_1 h_2 \ln \left( \frac{r_2}{r_1} \right) \ln \left( \frac{a_2}{a_1} \right)} \int_0^\infty S(kr_1, kr_2) S(ka_1, ka_2) \cdot Q(kh_1, kh_2) e^{-k|z|} \, dk$$  \hspace{1cm} (16)$$

A. Numerical Calculations

Equation (16) appears rather formidable, however, it is perfectly amenable to numerical evaluation.

In the past, the filament formula (2) has been used for coils with a filament placed at the center of the section and with $z$ replaced by the Geometric Mean Distance (GMD) between the coils [2]. In the case of self inductance, $z$ is replaced by the GMD of the coil from itself, $\text{GMD} = 0.2235 (w + h)$. The central filament is placed so that the current is divided equally on either side of the filament.

As a further improvement it seems reasonable that an equivalent filament could be obtained provided that the filament is so placed that the correct current density is taken into account. Integration of (6) shows that equal current division occurs at the radius given by the geometric mean (GM) of the inside and outside radii $[r_0 = \sqrt{(r_1 \cdot r_2)}]$.

There are three cases (see Fig. 3):

1) Self Inductance $L_1$

Replace $z$ in (3) by GMD of the coil from itself. Place filament at the center or at the GM of the cross-section,

2) Mutual Inductance $M_{12}, M_{13}, z \neq 0$

$z$ is replaced by the GMD between sections. For sections with different radial dimensions, such as 1 and 3 in Fig. 3, it is sufficiently accurate to take GMD = $z$. Place filament at the center or at the GM of the cross-section.

3) Mutual Inductance $M_{14}, z = 0$.

In this case a single filament is not sufficiently accurate. Lyle's Method [2] is used here with two filaments replacing each section. The radial dimensions are given by

$$r_{1,2} = R \left( 1 + \frac{h^2}{24R^2} \right) \pm \sqrt{\frac{w^2 - h^2}{12}}$$  \hspace{1cm} (17)$$

where $R$ is taken at the center or the GM of the cross-section.

The total mutual inductance between the two sections is the sum of the individual mutual inductances between the equivalent filaments of each section, each carrying half the total current.

$$M_{14} = (M_{ac} + M_{ad} + M_{bc} + M_{bd})/4$$  \hspace{1cm} (18)$$

where $a$ and $b$ represent the filaments in one cross-section and $c$ and $d$ represent the filaments in the other cross-section.

B. Experimental Validation

An experimental device was constructed with the dimensions shown in Fig. 3.

Table I summarizes the results for the following conditions:

1) Measurement; this was carried out at 10 kHz to avoid high frequency effects which shall be discussed later.
Fig. 3. Layout of experimental device.

TABLE I

<table>
<thead>
<tr>
<th>Measurement</th>
<th>L₁</th>
<th>M₂</th>
<th>M₃</th>
<th>L₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEA (ANSOFT)</td>
<td>4.39e+06</td>
<td>3.942</td>
<td>2.210</td>
<td>628</td>
</tr>
<tr>
<td>New Model (16)</td>
<td>4.365</td>
<td>3.851</td>
<td>2.229</td>
<td>627</td>
</tr>
<tr>
<td>GM Approximation</td>
<td>4.391</td>
<td>4.017</td>
<td>2.217</td>
<td>627</td>
</tr>
<tr>
<td>Center Approximation</td>
<td>4.431</td>
<td>4.145</td>
<td>2.267</td>
<td>615</td>
</tr>
</tbody>
</table>

L₁ is the self-inductance of the prototype device. All values are in H.
L₂ is the self-inductance of section 1 of figure 3 and M₁ is the mutual inductance between sections 1 and 2 etc. Clearly the GM approximation provides a very accurate estimate of the overall inductance. The approximation for filaments at the center of sections is quite good in this case, however, as the ratio of increases the error grows rapidly.

2) Finite Element Analysis; the finite element analysis (FEA) [9] was carried out at 10 kHz.
3) Model; Numerical evaluation of (16) [10].
4) GM Approximation; Equation (2) with filaments at the geometric means.
5) Center Approximation; Equation (2) with filaments at the center of sections.

C. High Frequency Effects

A general lumped-parameter model of the prototype device is shown in Fig. 4. The capacitance C consists of the three interlayer capacitances of the device connected in series. The input impedance of the device was measured on a HP network analyzer from 10 kHz to 100 MHz and the results are shown in Fig. 5. The input impedance of the equivalent circuit of Fig. 4 was calculated for L = 628 nH as calculated, R = 15.7 Ω and C = 10.8 pF. The resonant frequency is 61 MHz. The capacitance C can be estimated by calculating the interlayer capacitance in Fig. 3 and treating each layer as a parallel plate capacitor, the dielectric constant of the insulating material is 7. This gives a predicted value of C = 13.4 pF which in turn gives a resonant frequency of 55 MHz. In the prototype device the turns are spiraled and do not overlap in exact concentric circles, however, a reasonable estimate of resonant frequency is obtainable using this straightforward approach. The equivalent resistance is a function of the skin effect losses and proximity effect losses and the effective ac resistance can be found from finite element analysis simulation of the device [9].

III. Spiral Coil on a Ferromagnetic Substrate

The presence of a ferromagnetic substrate in the vicinity of the planar coil in Fig. 2 enhances its self inductance. If the half plane (z < 0) were replaced by an ideal magnetic material (σ = 0, μ₁ = ∞) the self inductance would be doubled as compared with the air case [4]. The presence of coil currents gives rise to eddy current effects in a ferrite with finite conductivity. The ultimate application of these devices necessarily means high frequency operation and therefore a general impedance equation is required, which takes frequency dependent eddy current losses in the substrate into account. In this section such a generalized impedance formula is derived for magnetic substrates, similar to (16) in Section II, which takes full account of eddy currents in the substrate. The starting point of the analysis is the mutual impedance between two filaments placed above a magnetic substrate as shown in Fig. 6. The substrate is assumed to be infinite in the −z direction. In practice, the substrate should be at least five skin depths thick to ensure the validity of this assumption. The lower filament in Fig. 6 is at a height d above the substrate, so that a dielectric layer can be accounted for later.

Maxwell's Equations are solved from first principles for the configuration in Fig. 6 and the details are given in the
Appendix. The mutual impedance between the two filamentary circular concentric turns of Fig. 6 is

\[
Z = j \omega M + Z_s
\]

(19)

where \( M \) is the mutual inductance which would exist in the absence of the substrate and is the same as (1). \( Z_s \) is the additional impedance due to the presence of the substrate.

\[
Z_s = R_s + j \omega L_s = j \omega \mu_0 \pi a r \int_0^\infty J_1(kr)J_1(ka) \phi(k) e^{-k(d_1+d_2)} dk
\]

(20)

\[
\phi(k) = \frac{\mu_r - \eta}{\mu_s + \eta} \frac{k}{k}
\]

(21)

\[
\eta = \sqrt{k^2 + j \omega \mu_0 \mu_s \sigma}
\]

(22)

A filament placed directly on an ideal magnetic substrate (\( d = 0, \sigma = 0, \mu_r \rightarrow \infty \)) means \( \eta = k \) and \( L_s = M \) giving a doubling of the inductance as expected. In air \( \mu_r = 1 \) and \( \Phi(k) = 0 \) giving \( L_s = 0 \) as expected. Fig. 7 shows two circular concentric planar sections on a magnetic substrate. The current density is taken as being inversely proportional to the radius as described in Section II. Applying the procedure outlined in Section II to the substrate term \( Z_s \) in (20) gives

\[
Z_s = \frac{j \omega \mu_0 \pi}{h_1 h_2} \ln \frac{r_2}{r_1} \ln \frac{a_2}{a_1} \int_0^\infty S(kr_2, kr_1) S(ka_2, ka_1)
\]

\[
\cdot Q(kh_1, kh_2) \phi(k) e^{-k(d_1+d_2)} dk.
\]

(23)

**A. Validation of the New Formula**

The frequency dependent mutual impedance formula (23) takes full account of eddy current losses in the substrate. The resistive component of \( Z_s \) represents the substrate losses and the reactive component of \( Z_s \) represents the enhanced inductance due to the reflected field of the magnetic substrate.

Simulations were carried out for the device shown in Fig. 3 on a magnetic substrate of transformer steel (\( \sigma = 2 \times 10^6 (\Omega \cdot m)^{-1}, \mu_r = 1000, d = 0 \)) using finite element analysis. The self impedance results are shown in Fig. 8. The calculated results were obtained using the impedance formula (23) in conjunction with (16) to account for the air term. The dc resistance of the coil (0.686 \( \Omega \)) is included in the calculated resistance. Skin and proximity effects in the winding are not included. There is very good agreement between the simulated and calculated results which establishes the validity of the proposed formula in predicting the effect of a magnetic substrate on the inductance and on the losses in a planar magnetic device. It is noteworthy that at 40 MHz skin and proximity effect losses contributed less than 5% to the total losses, evidently at very high frequencies the eddy current losses in the substrate dominate. The most salient feature of Fig. 8 is that the inductance remains essentially flat up to 1 MHz. Clearly with a less lossy substrate such as ferrite (\( \sigma = 1 (\Omega \cdot m)^{-1} \)) the frequency where the inductance falls off is several orders of magnitude above 1 MHz.

**IV. Conclusions**

A new set of formulas has been established for calculating self and mutual impedances of planar coils on homogeneous ferromagnetic substrates. The formula for planar spiral coils in air is a special case of the general formula.
The formulas have been derived from Maxwell’s equations and therefore they can be fully expected to represent practical planar devices accurately. As such, the formulas serve as a useful gauge for simpler approximations. While simple approximations are useful in initial design, it should be borne in mind that mathematical software packages have reached such a maturity that they are reliable, readily available and straightforward to use.

Comparisons between experimental and calculated data, for a 4 layer 12 turn device, show that the new formula represents physical planar devices accurately. The work presented in this paper lays the groundwork for future developments, such as sandwich inductors where another substrate is added above the coil winding.

V. ACKNOWLEDGMENT

The assistance of the National Microelectronic Research Center, University College, Cork and Pulse Engineering, Tuam, Co. Galway, Ireland, is gratefully appreciated.

APPENDIX

For a magnetothermal system, the following forms of Maxwell’s Equations hold in a linear homogeneous isotropic medium

\[
\mathbf{\nabla} \times \mathbf{H} = \mathbf{J}_\phi
\]

\[
\mathbf{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.
\]  

(A1)

The filamentary turn at \(z = d_1 = d\), in Fig. 6, carries a sinusoidal current \(i_b(t) = I_b e^{i\omega t}\). Medium 1 refers to \(z > 0\) in air and medium 2 refers to the magnetic substrate \(z < 0\).

On the basis of cylindrical symmetry the following identities apply to the electric field intensity \(E\) and the magnetic field intensity \(H\)

\[
E_r = 0, \quad E_z = 0, \quad \frac{\partial E_\phi}{\partial \phi} = 0
\]

\[
H_\phi = 0, \quad \frac{\partial H_z}{\partial \phi} = 0, \quad \frac{\partial H_r}{\partial \phi} = 0.
\]  

(A2)

Maxwell’s Equations reduce to Medium 1 (\(z \geq 0\)):

\[
\frac{\partial H_\phi}{\partial z} - \frac{\partial H_z}{\partial r} = I_b \delta(r - a) \delta(z - d)
\]

\[
\frac{\partial E_\phi}{\partial z} = j\omega \mu_0 H_r,
\]

\[
\frac{1}{r} \frac{\partial (rE_\phi)}{\partial r} = -j\omega \mu_0 H_z.
\]  

(A3)

Medium 2 (\(z \leq 0\))

\[
\frac{\partial H_\phi}{\partial z} - \frac{\partial H_z}{\partial r} = \sigma E_\phi
\]

\[
\frac{\partial E_\phi}{\partial z} = j\omega \mu_\epsilon H_r,
\]

\[
\frac{1}{r} \frac{\partial (rE_\phi)}{\partial r} = -j\omega \mu_\epsilon H_z.
\]  

(A4)

\(E\) has a \(\phi\)-component only and we shall drop the \(\phi\) subscript. Eliminating \(H\) gives the following result for medium 1

\[
\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} = \frac{E}{r^2} + j\omega \mu_0 I_b \delta(r - a) \delta(z - d).
\]  

(A5)

Applying the Fourier-Bessel Integral Transform [11], noting that:

\[
\int_0^\infty \delta(r - a)J_1(kr)r \, dr = aJ_1(ka)
\]

gives the transformed version of (A5)

\[
\frac{d^2 E^*}{dz^2} = k^2 E^* + j\omega \mu_0 I_b J_1(ka) \delta(z - d).
\]  

(A6)

Equation (A6) has a solution of the form

\[
E^* = Ae^{-\xi z} + Be^{\xi z}.
\]  

(A7)

At this point we must distinguish between the region of medium 1 above the filament i.e. \(z > d\) and the area between the filament and the substrate i.e. \(0 < z < d\). Clearly for \(z > d\), the field decays at infinity and \(E^*\) is given by

\[
E^* = Ae^{-\xi z} \quad z \geq d\]

\[
E^* = Be^{\xi z} + Ce^{-\xi z} \quad 0 \leq z \leq d.
\]  

(A8)

Eliminating \(H\) in (A3) gives

\[
\frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} - \frac{E}{r^2} = j\omega \mu_0 \sigma E.
\]  

(A9)

Applying the Fourier-Bessel Integral Transform

\[
\frac{d^2 E^*}{dz^2} - (k^2 + j\omega \mu_0 \sigma) E^* = 0.
\]  

(A10)

The electric field in medium 2 must tend to zero at infinity \((z \to \infty)\) and therefore the solution of (A10) is of the form

\[
E^* = De^{\eta z}.
\]  

(A11)

where

\[
\eta = \sqrt{k^2 + j\omega \mu_0 \sigma}.
\]

The following boundary conditions apply:

(a) \(E\) is continuous at the boundary between the two media at \(z = 0\)

\[
B + C = D
\]  

(A12)

(b) The radial component of \(H\) is continuous at the boundary between the two media at \(z = 0\) From (A1)
\[
\frac{\partial E}{\partial z} = j\omega \mu_0 \mu_H, \quad (A13)
\]

Application to (A7) and (A11), noting that \(\mu_r = 1\) in medium 1 with rearranging gives
\[
D = \mu_r \frac{k}{\eta} (B - C) \quad (A14)
\]
(c) \(E\) is continuous in the plane of the filament at \(z = d\)
\[
A e^{-kd} = B e^{kd} + C e^{-kd} \quad (A15)
\]
(d) The boundary condition on the \(H\) field in the plane of the filament at \(z = d\) is given by
\[
\vec{n} \times (\vec{H}_z - \vec{H}_-) = K_f
\]
where \(\vec{n}\) is the unit vector normal to the plane of the filament and \(K_f\) is the surface current density at the boundary which is given by
\[
K_f = \int_0^d \left. \vec{I}_0 \delta(r - a) \delta(z - d) \right. \, dz = \vec{I}_0 \delta(r - a)
\]
and in terms of transformed variables
\[
K_f = \vec{I}_0 a J_1(ka)
\]
\(H_z\) and \(H_-\) are found from (A13) and the boundary condition now becomes
\[
-k A e^{-kd} - k (B e^{kd} - C e^{-kd}) = j\omega \mu_0 \vec{I}_0 a J_1(ka). \quad (A16)
\]
The four unknowns \(A, B, C\) and \(D\) are now solved using (A12), (A14), (A15) and (A16) to give \(E^*\) in medium 1 for \(z > d\)
\[
E^* = -\frac{j\omega \mu_0 \vec{I}_0 a}{2k} \left[ e^{-k(z - d)} + \phi(k) e^{-k(z + d)} \right] J_1(ka)
\]
(A17)

where
\[
\phi(k) = \frac{\mu_r - \frac{\eta}{k}}{\mu_r + \frac{\eta}{k}}.
\]

Applying the inverse transform of the Fourier-Bessel Integral
\[
E = -\frac{j\omega \mu_0 \vec{I}_0 a}{2k} \int_0^\infty \left[ e^{k \mp d} + \phi(k) e^{-k \mp d} \right] J_1(ka) J_1(kr) \, dk.
\]
(A19)

An alternative derivation of (A19) using the method of separation of variables is given in reference [12].

Mutual impedance \(Z\), between the source at \((a, d)\) and the circular filament at \((r, d)\) gives the induced voltage
\[
V = Z I_0. \quad (A20)
\]

Thus
\[
Z = -\frac{2\pi r E(r, d)}{I_0} - (A21)
\]

The mutual impedance is
\[
Z = j\omega M + Z_i \quad (A22)
\]
where \(M\) is the component of mutual inductance which would exist in the absence of the substrate and corresponds to (1) and
\[
Z_i = j\omega \mu_0 \pi a r \int_0^\infty J_1(kr) J_1(ka) \phi(k) e^{-k(d + d)} \, dk \quad (A23)
\]

REFERENCES


William Gerard Hurley (M’77, SM’90) was born in Cork, Ireland in 1952. He graduated from the National University of Ireland, Cork in 1974 with a first class honors degree in Electrical Engineering. He received the Master’s degree in electrical engineering at the Massachusetts Institute of Technology in 1976. He received the Ph.D. in Transformer Modelling at the National University of Ireland, Galway in 1988. He worked for Honeywell Controls in Canada as a product engineer from 1977 to 1979 and as a development engineer in transmission lines at Ontario Hydro from 1979 to 1983. He lectured in electronic engineering at the University of Limerick, Ireland from 1983 to 1991 and is currently a senior lecturer in the Department of Electronic Engineering at University College, Galway, Ireland. He is the director of the Power Electronics Research Center there.

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