Effects of aberrations on transfer functions used in high angular resolution astronomical imaging

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Abstract. In this paper we present the results of the effect of aberrations on the transfer functions used in the high-angular resolution astronomical imaging techniques of speckle interferometry, Knox-Thompson and bispectral imaging. The analyses are based on a computer simulation of imaging through atmospheric turbulence. The results show that as the seeing becomes worse, its effect dominates the behaviour of the transfer functions which tend to be independent of (small) optical aberrations. However, if the wavefront variation due to fixed aberrations is significant over $r_0$-sized regions in the pupil (where $r_0$ is the Fried parameter), the above transfer functions do depend on the aberration: in particular, the bispectral transfer function is relatively sensitive to odd aberrations, such as coma.

1. Introduction

The effect of optical aberrations on transfer functions in high-resolution imaging in astronomy has been studied by many scientists since Labeyrie invented the technique of speckle interferometry in 1970 [1]. Speckle interferometry yields a diffraction-limited estimate of the Fourier modulus of the object (or equivalently, the autocorrelation of the object intensity), whereas extensions of it, in particular Knox-Thompson [2, 3] and bispectrum imaging [4, 5], provide an estimate of the Fourier phase as well. It is clearly important to know in all three cases the extent to which the aberrations (including defocus) affect the transfer function of the imaging process.

Early investigations [6–9] showed that the speckle transfer function was much less sensitive to telescope aberrations than the normal transfer function applicable in the absence of turbulence. Telescope aberrations only begin to affect the speckle transfer function when the wavefront error is significant over a linear dimension comparable to the Fried parameter $r_0$: this means that, as the seeing deteriorates (i.e. $r_0$ decreasing), the speckle transfer function becomes less sensitive to aberrations. Some experimental results on the effect of defocus were given by Karo et al. [10] and computational results, also on the effect of defocus, by Roddier et al. [11].

Two detailed studies of the effect of aberrations on the transfer functions in speckle interferometry [12], Knox-Thompson imaging [12], and bispectrum imaging [13] have been made. In both cases, the atmospheric turbulence was modelled with a Gaussian correlation function. Publications of other authors who describe the effect of aberrations can be found in references [14–16].

If the fixed wavefront aberration is so severe that path length fluctuations greater than the coherence length of the light are introduced, then it may be necessary to reduce the bandwidth and thus increase the coherence length: this will clearly have an effect on the signal-to-noise ratio. This effect is unlikely to be encountered in practice and is not discussed here.
In contrast to previous work, we use a Monte Carlo computer simulation method to investigate the effect of telescope aberrations on the three transfer functions. This involves the generation of a large number of short-exposure images of a point source in the computer and their analysis in terms of the power spectrum, cross-spectrum and bispectrum. Although this is a fairly clumsy method of analysis, it has proved to be productive and has revealed the crucial limitations which result when a finite number of short-exposure images are used.

2. General theory of transfer functions

All speckle interferometric imaging techniques operate by processing a large set of short-exposure speckle images (interferograms) in a certain way. The actual behaviour of a transfer function of a specific method depends on the way in which the images are processed. For each instantaneous record, it is assumed that the usual quasi-monochromatic isoplanatic imaging equation applies:

\[ I(x, y) = O(x, y) \ast P(x, y), \]

(1)

where \( I(x, y) \) is the instantaneous image intensity as a function of coordinate variables \( x \) and \( y \), \( O(x, y) \) is the object intensity, \( P(x, y) \) is the instantaneous point spread function of the atmosphere/telescope system normalized to unit volume, and \( \ast \) denotes the convolution integral. In the Fourier domain the equivalent expression is:

\[ i(u, v) = o(u, v) \ast p(u, v), \]

(2)

where \( i(u, v) \), \( o(u, v) \) and \( p(u, v) \) are Fourier transforms of \( I(x, y) \), \( O(x, y) \) and \( P(x, y) \). For the purpose of studying transfer functions, an unresolved star will be used for imaging. This means the object intensity \( O(x, y) \) is a delta-function \( \delta(x, y) \) and its Fourier transform is a constant. Then it follows that \( i(u, v) = p(u, v) \), where \( p(u, v) \) is the instantaneous modulation transfer function of the incoherent optical system, and equals the autocorrelation of the pupil function [17]:

\[ p(u, v) = \int \int_{-\infty}^{+\infty} H(x_1, y_1) H^*(x_1 + x, y_1 + y) \, dx_1 \, dy_1, \]

(3)

where \( H(x, y) \) is the pupil function. The variables \( x \) and \( y \) represent distance in the pupil and are related to the spatial frequency \( u \) and \( v \) by \( x = f \lambda u \) and \( y = f \lambda v \), where \( \lambda \) is the wavelength and \( f \) is the focal length.

In the case of imaging through a random medium with an aberrated instrument, the pupil function \( H(x, y) \) may be split into a product of two functions:

\[ H(x, y) = H_0(x, y) A(x, y), \]

(4)

where \( H_0(x, y) \) is real, representing the telescope aperture alone and \( A(x, y) \) is the aberration function of the form:

\[ A(x, y) = \exp [ikW(x, y)] Z(x, y), \]

(5)

where \( k = 2\pi/\lambda \) is the wavenumber, \( W(x, y) \) is the deterministic part due to the aberrations of the optical system and \( Z(x, y) \) is the random part induced by the turbulent atmosphere. It is these two parts that will dominate the actual form of the
transfer function. Substitution of equation (4) and equation (5) into equation (3) yields:

\[ p(u, v) = \int \int_{-\infty}^{\infty} H_0(x_1, y_1) H_0(x_1 + x, y_1 + y) \exp \{ik[W(x_1, y_1) - W(x_1 + x, y_1 + y)]\} Z(x_1, y_1) Z^*(x_1 + x, y_1 + y) \, dx_1 \, dy_1. \]  

(6)

The forms of the deterministic idealized aberrations \( W(x, y) \) are:

Defocus

\[ W(x, y) = \frac{\Delta f}{2f^2} (x^2 + y^2), \]  

(7)

Spherical aberration

\[ W(x, y) = \frac{1}{8} S_1 (x^2 + y^2)^2, \]  

(8)

Coma

\[ W(x, y) = \frac{1}{8} S_2 (x^2 + y^2)y, \]  

(9)

Astigmatism

\[ W(x, y) = \frac{1}{8} S_3 y^2, \]  

(10)

where \( \Delta f \) is the defocus value and \( S_1, S_2 \) and \( S_3 \) are the Seidel coefficients for spherical aberration, coma and astigmatism, respectively [18].

The calculation of the transfer functions requires a suitable statistical model for the atmospheric turbulence. Korff [19] and Roddier et al. [11] use the log normal model for the complex amplitude, which is the most realistic for typical seeing and will be adopted in this work. Then \( Z(x, y) \) can be written in terms of the log amplitude \( l(x, y) \) and phase \( \phi(x, y) \), both of which are each assumed to have Gaussian or normal distributions and be statistically independent, which is approximately true in practice [20]:

\[ Z(x, y) = \exp [l(x, y) + i\phi(x, y)]. \]  

(11)

Equation (6), in which \( p(u, v) \) is a random process because of the effect of atmospheric turbulence, is a general form of the instantaneous modulation transfer function of the whole imaging system. All imaging transfer functions in astronomy are based on this equation. The transfer functions of concern here are: the speckle transfer function (STF) \( p^{(\text{STF})}(u, v) \), the bispectral transfer function (BTF) \( p^{(\text{BS})}(u_1, u_2) \) and the Knox-Thompson transfer function (KTF) \( p^{(\text{KT})}(u, \Delta u) \).

The speckle transfer function, bispectral transfer function and Knox-Thompson transfer functions are defined by

\[ p^{(\text{STF})}(u, v) = \langle |p(u, v)|^2 \rangle \]  

(12)

\[ p^{(\text{BS})}(u_1, u_2) = \langle p(u_1) p(u_2) p^*(u_1 + u_2) \rangle, \]  

(13)

\[ p^{(\text{KT})}(u, \Delta u) = \langle p(u) p^*(u + \Delta u) \rangle. \]  

(14)
3. Computer simulation

\( p^{(BS)}(u, v) \) is a two-dimensional function that can be calculated and displayed in a straightforward way. However, both \( p^{(BS)}(u_1, u_2) \) and \( p^{(KT)}(u, \Delta u) \) are four-dimensional functions and their calculations (and display) is non-trivial. For this reason, the computer simulations for these two cases are one-dimensional: we believe that the one-dimensional results reproduce all the essential features of the two-dimensional case and are considerably easier to interpret.

Each image is Fourier transformed to get the instantaneous transfer function \( p(u, v) \) or \( p(u) \), from which the desired transfer function is calculated; a large number \( N \) of these individual transfer functions are summed together to give an estimate of the ensemble average. Typically, one thousand frames were processed in the two-dimensional case (speckle transfer function) and ten thousand frames in the one-dimensional cases (bispectrum and Knox–Thompson transfer functions).

The power spectrum of the random wavefront is taken to be Kolmogorov [21]:

\[
W(f) = \frac{0.023}{r_0^{5/3} f^{11/3}},
\]

where \( f \) is spatial frequency. To create random wavefronts with the power spectrum given by equation (15), the method introduced by McGlamery [23] was used. This does not provide a good simulation of the very low spatial frequencies in the instantaneous image [22] (i.e. the image motion) but this is not important in the present case.

Three seeing values \( r_0 \) are used: 20 cm, 10 cm and 5 cm. For a 2.5 m telescope with a 0.914 m central obscuration, the corresponding speckle numbers within a seeing disc defined by \( N_s = 2.3 (D/r_0)^2 \) are 360, 1440 and 5750, respectively. When one-dimensional data is generated, the aperture diameter is set to 2 m without a central obscuration for the bispectrum and 2.5 m for the Knox–Thompson analysis.

4. Speckle transfer function: results and discussion

Results of the calculation of the speckle transfer function under different seeing combined with various kinds of aberrations are plotted in figures (1–4) and figure 6. In all cases, the aberration-free curve is presented along with aberrated transfer function curves for comparison; vertical axis is the normalized log \( 10 \) value of the speckle transfer function (STF) and horizontal axis is the spatial frequency relative to the diffraction-limited cut-off frequency. In each figure, the top graphs (a) are for good seeing \( (r_0 = 20 \text{ cm}) \), the middle graphs (b) for average seeing \( (r_0 = 10 \text{ cm}) \) and the lower graphs (c) for poor seeing \( (r_0 = 5 \text{ cm}) \).

A comparison of the aberration-free STFs in different seeing conditions (e.g. solid curves in figure 1 (a, b and c)), shows the well-known result that the speckle transfer function is very sensitive to the seeing parameter \( r_0 \), with the mid-frequency plateau having a value \( \approx 1/N_s \), where \( N_s \) is the average number of speckles defined above. Bearing in mind the fact that the maximum permissible bandwidth and exposure time also decrease with decreasing \( r_0 \), then it can be seen that the signal-to-noise ratio of an estimate of the object power spectrum in speckle interferometry deteriorates very rapidly in poor seeing.

The broad conclusion of figures 1–4, which show the effect of defocus, spherical aberration, coma and astigmatism on the speckle transfer function, is that the effect of aberrations is less when the seeing is poor. The magnitude of the effect of
Figure 1. Speckle transfer functions in the presence of defocus for (a) $r_0 = 20$ cm, (b) $r_0 = 10$ cm and (c) $r_0 = 5$ cm. The amounts of defocus (from the paraxial focus) are 0, 0.5 mm, 1.0 mm and 2.0 mm for an f/15 telescope of diameter 2.5 m with a central obstruction of diameter 0.9 m. As the seeing deteriorates, the speckle transfer functions become independent of any aberration.

Figure 2. As figure 1, except for spherical aberration values of 0, $\lambda$, $2\lambda$ and $4\lambda$, where the aberration is measured at the maximum pupil diameter.
Figure 3. As figure 1, except for coma values (sagital and tangential) of 0, \(\lambda\), 2\(\lambda\) and 4\(\lambda\), where the aberration is measured at the maximum pupil diameter.
Effects of aberrations

Figure 4. As figure 1, except for astigmatism of 0, λ, 2λ and 4λ, where the aberration is measured at the maximum pupil diameter.

Figure 5. The effect of defocus and spherical aberration on the conventional diffraction limited optical transfer functions (OTF) for the same telescope parameters as figure 1. Note that the magnitude of aberration considered has a very large effect on the OTF.

aberrations is not large, at least compared to the effect of the same aberrations on a diffraction-limited imaging system (with no atmospheric turbulence). This is illustrated in figure 5, which shows the effects of defocus and spherical aberration on the same optical system, but diffraction-limited with no atmospheric turbulence. For example, 2 mm of defocus has a catastrophic effect on the normal transfer function in the absence of turbulence, whereas the effect on the speckle transfer function is negligible in typical seeing (r0 = 10 cm). According to geometrical optics, a 2 mm defocus in an f/15, 2.5 m diameter telescope corresponds to a 0.73 arc s diameter blur: a 2λ value for spherical aberration corresponds to a blur of ≈1.45 arc s.
Figure 6 illustrates that the principle of aberration-balancing, used routinely in the design of optical systems for normal imaging does not apply in precisely the same way to power spectrum transfer functions. The reason for this is that, in speckle interferometry, the transfer function is insensitive to aberrations provided that the aberrations are essentially constant (say $<\lambda/4$) over each patch in the pupil of diameter $r_0$: in particular, the relative phase disturbance due to the aberration between the centre and the edge of the pupil is unimportant (provided of course it is less than the coherence length). In normal imaging, the latter factor is crucial, and the balancing of aberrations allows this phase difference to be minimized.

5. Bispectral transfer functions: results and discussion

The calculations for both the bispectral and Knox-Thompson transfer functions were carried out for an unobstructed one-dimensional pupil, corresponding to an $f/15$ telescope of diameter 2 m or 2.5 m.

It is widely recognized, following the detailed argument of Lohmann et al. [5] and the phase closure argument of Roddier [24], that the bispectral transfer function should have zero phase. However, there is anecdotal evidence from those making observations that telescope aberrations do affect the phase of the bispectral transfer function and accordingly we concentrate our study on the bispectral phase.

Figure 7 shows phase maps of aberration-free bispectral transfer functions (BTFs) averaged using (b) $10^3$, (d) $10^4$ and (f) $10^5$ frames. However, such finite averaging leaves a significant phase variance $\sigma_1^2$, which is defined by:

$$\sigma_1^2 = \sigma_F^2 \sin^2(\Phi) + \sigma_R^2 \cos^2(\Phi) - \text{cov}(R, I) \sin(2\Phi),$$

(16)
Figure 7. Bispectral error (left) and averaged bispectral phase (right) for $D/r_0=10$ with no fixed aberrations, as determined over $10^3$, $10^4$ and $10^5$ realizations. A significant residual average phase may remain if the number of realizations used is too small.
Figure 8. As figure 7, but for 2 mm defocus in an $f/15$ telescope of diameter 2 m and no central obstruction.
Figure 9. Modulus and phase of the bispectral transfer function in the absence of any fixed aberrations, for three values of $r_0$ (20, 10 and 5 cm) for a 2 m telescope with an unobstructed pupil. The hexagonal support indicates the diffraction-limited frequencies.
Figure 10. As figure 9, but for defocus values of 0, 1 mm and 2 mm (for an f/15 system) and $r_0 = 20$ cm (good seeing).
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Figure 11. As figure 9, but for defocus values of 0, 1 mm and 2 mm (for an f/15 system) and $r_0=10$ cm (average seeing).
Figure 12. As figure 9, but for defocus values of 0, 1 mm and 2 mm (for an f/15 system) and $r_o=5$ cm (poor seeing).
Figure 13. As figure 9, but for spherical aberration of 0, λ and 2λ for $r_0 = 10$ cm.
Figure 14. As figure 9, but for coma of 0, 0.5λ and λ for $r_0 = 10$ cm. Note the significant residual average phase close to the axes of symmetry.
where $\sigma_R^2$ and $\sigma_I^2$ are variances of the real and imaginary components of $p^{TC}(u_1, u_2)$; $\text{cov} \ (R, I)$ is the covariance of the real and imaginary components of $p^{TC}(u_1, u_2)$ and $\Phi$ is the argument of $p^{TC}(u_1, u_2)$, i.e. the phase of the BTF. If $M$ is the total number of independent realizations used for calculating the BTF, an estimate of the phase error, $\theta_e(u_1, u_2)$, can be obtained using:

$$\theta_e(u_1, u_2) = \arctan \left( \frac{\sigma_I}{|p^{TC}(u_1, u_2)| M^{1/2}} \right).$$  \hspace{1cm} (17)

Figures 7 \((a, c \text{ and } e)\) show the phase errors in the aberration-free case resulting from the finite numbers of realizations. Clearly, even in the aberration-free case, the effect of finite averaging can leave a significant residual phase in the bispectral transfer function. This effect is larger in the presence of aberrations, as shown in figure 8 for the case of 2 mm of defocus (very severe). It is therefore essential, both in this study and of course when making real measurements, to carry out the averaging over a sufficiently large ensemble. It is interesting to note that an important application of bispectral techniques may be to satellite imaging, where it is usually possible to

Figure 15. Modulus and phase of the Knox–Thompson transfer function in the absence of any fixed aberrations, for two values of $r_0$ (20, 10 cm) for a 2.5 m telescope with an unobstructed pupil. The parallelogram support indicates the diffraction-limited frequencies.
gather only a few hundred frames at most due to motion of the satellite. In the following, both the modulus and the phase of BTFs averaged over 10000 realizations will be presented.

Figure 9 shows the effect of seeing on the aberration-free bispectral transfer function. The modulus decreases gracefully as the seeing gets poorer: in fact, most algorithms for image recovery from the bispectrum use the modulus estimate given by the power spectrum, so the behaviour of the modulus of the BTF is somewhat academic. The bispectral phase is essentially zero in good seeing but departs significantly from zero when \( r_0 = 5 \text{ cm} \) \((D/r_0 = 40)\). Taking a larger number of realizations will decrease this phase to effectively zero but the number used in the simulation—\(10^4\)—probably represents a realistic upper bound to observational practice. Thus good seeing is desirable for bispectral imaging for this reason (in addition to the usual signal-to-noise ratio advantages).

Figures 10, 11 and 12 show the effect of defocus for the three seeing conditions of \( r_0 = 20 \text{ cm}, 10 \text{ cm} \) and \( 5 \text{ cm} \). It is difficult here to separate the effects of finite averaging due to increasing aberration and/or deteriorating seeing. However, it is clear that in good seeing (figure 10), the bispectral transfer function is much more sensitive to defocus than in poor seeing (figure 12): in fact, for the parameters of figure 12, the defocusing has little influence on the BTF. It should be emphasized that the amounts of defocus considered far exceed anything that should be encountered in practice.

For the remaining aberrations, we consider only the case of typical seeing, \( r_0 = 10 \text{ cm} \), in order to reduce the data to manageable proportions (all cases have in fact been computed). Figures 13 and 14 show the effects of spherical aberration and coma. The main feature to point out here is the residual phase of the BTF close to important lines of symmetry, for the case of coma: these are the regions of high signal-to-noise ratio in the bispectrum used in object reconstruction algorithms. This sensitivity to coma was also reported by Barakat and Ebstein [13].

6. Knox-Thompson transfer functions: results and discussions

The Knox-Thompson technique [2, 3] of speckle imaging is very similar to the bispectral method: a detailed comparison is given in [25]. However, the closure phase argument [24] is not applicable in this case and therefore one would expect an increased sensitivity to abberations compared to the bispectral technique: this is broadly confirmed by our calculations. In the calculations presented here, all frequency offset vectors \( \Delta u \) are displayed (i.e. the generalized Knox-Thompson technique [25]).

Aberration-free Knox-Thompson transfer functions (KTF) for \( r_0 = 20 \text{ cm} \) and \( 10 \text{ cm} \) are shown in figure 15, for \( 10^4 \) realizations. It is clear from the maps of the modulus of the KTF (left hand column) that there is a narrow strip of values of the frequency offset vector \( \Delta u \) for which the KTF is non-zero, and along this strip it approximately equals the speckle transfer function. The strip is clearly narrower in poorer seeing. Correspondingly, in the phase transfer functions, one sees that the 'zero phase' strip has the same width as that of the modulus.

Figures 16–18 show the effect of defocus, spherical aberration and coma on the Knox-Thompson transfer function, for \( r_0 = 10 \text{ cm} \) and \( 10^4 \) realizations. The phase of the KTF quickly degrades with increasing aberration and the effects of aberration is clearly greater than in the case of the bispectral transfer function.
Figure 16. As figure 15, but for defocus values of 0, 1 mm and 2 mm (for an f/15 system) and $r_0 = 10$ cm (average seeing).
Figure 17. As figure 15, but for spherical aberration of 0, $\lambda$ and $2\lambda$ for $r_0 = 10\,\text{cm}$. 
Figure 18. As figure 15, but for coma of 0, 0.5λ and λ for $r_0 = 10$ cm. Note the significant residual average phase along the vertical axis.
7. Conclusions

Compared to conventional imaging, in which the important transfer function is the usual optical transfer function, all three methods considered here are significantly less sensitive to telescope aberrations. For the speckle transfer function, aberrations are not important unless they vary over regions comparable to the seeing parameter $r_0$—this implies a very large amount of aberration.

The phase of the bispectral transfer function is not exactly zero in every frame, as the principle of phase closure would imply, but is effectively zero on average in the aberration-free case if a very large number of realizations is used: however, this number may be larger than that possible in certain practical cases and there would then be a residual phase in the BTF. The effect of finite averaging is more severe when aberrations are present, and it appears that odd aberrations such as coma have a greater influence on the bispectrum transfer function. The Knox–Thompson transfer function is more sensitive to aberration than the bispectrum transfer function.

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